Filtration Safe Operations on Frames

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Abstract

Filtration is a standard tool for establishing the finite model property of modal logics. We consider logics and classes of frames that admit filtration, and identify some operations on them that preserve this property. In particular, the operation of adding the inverse or the transitive closure of a relation is shown to be safe in this sense. These results are then used to prove that every regular grammar logic with converse admits filtration. We present filtration constructions for right-linear and left-linear grammar logics. We also give a simple example of a grammar modal logic that is undecidable and hence does not admit filtration.

Keywords: Modal logic, tense logic, finite model property, filtration, transitive closure, universal modality, grammar modal logic, Horn closure, regular grammar, propositional dynamic logic.

Introduction

Filtration is a method of collapsing an infinite model into a finite one while preserving the truth values of a given finite set of formulas. In modal logic, it is widely used as a tool for establishing the finite model property (FMP) and decidability. It dates back to the pioneering works of Scott, Lemmon [15] and Segerberg [18]. This technique was used by Fischer and Ladner who designed

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a filtration for **PDL** [7], Shehtman who developed a filtration for products of modal logics (see e.g. [20]), and many others.

Gabbay [9] was perhaps the first who introduced the term "*a logic* **L** *admits filtrations*", which means that any model over an **L**-frame can be "filtrated" into a finite model again over an **L**-frame. He summarized that the logics **K**, **T**, **B**, **S4**, **S5**, **S4.1** admit filtration, and extended this list with the logics $\mathbf{K} + \Box p \to \Box^m p$, $m \geq 2$; the latter are the simplest examples of the so called *regular grammar logics* [5].

Our aim here is to investigate modal logics and classes of Kripke frames that admit filtration (or have the *AF property*), from the following viewpoint. We tackle the problem of identifying the cases when it is possible to transfer the AF property from a modal logic (or a class of frames) to its enrichment with new kinds of modalities (or accessibility relations).

Specifically, given a class $\mathcal{F}$ of frames of the form $F = (W, (R_e)_{e\in\Sigma})$, we consider the corresponding class $\mathcal{F}^{\circledast} = \{F^{\circledast} \mid F \in \mathcal{F}\}$ of their expansions $F^{\circledast} = (W, (R_e)_{e\in\Sigma}, S)$, where the relation $S \subseteq W \times W$ is obtained by an operation $\circledast$ on (the relations in) F. We investigate for which operations $\circledast$, the AF property for $\mathcal{F}$ implies that for $\mathcal{F}^{\circledast}$; we call such operations *filtration safe*.

This question has already been addressed before. In particular, Goranko and Passy [11] proved that adding the universal relation is filtration safe. Bezhanishvili and ten Cate [1] proved that taking the hybrid companion of a logic is filtration safe in [8], the AF property was applied to products of modal logics.

In general, the AF property is not only a tool for obtaining the FMP, but also a strong sufficient condition for the decidability of many derived modal logics, and so we believe it is worthy of studying per se. In particular, when we prove the AF property for some logic for which the decidability and FMP had already been established by other methods, this still increases our knowledge, because it allows us to make conclusions about the derived logics.

In this paper, we identify several frame expanding operations that preserve the AF property, namely: adding the union $R_a \cup R_b$, the composition $R_a \circ R_b$, the diagonal relation $\{(x, x) \mid x \in W\}$, and most interestingly – the inverse relation R_a^{-1} and the transitive closure relation R_a^+. Consequently, if a modal logic **L** admits filtration, then so does its tense counterpart $\mathbf{L}^t$, or extension with the transitive closure modality $\mathbf{L}^{\boxplus}$ (despite being "extremely dangerous" in general, cf. [2, p. 373]), provided that the logics obtained are Kripke complete. The result about $\mathbf{L}^t$ seems particularly interesting to us in context of Wolter's program of "temporalization" of modal logics [23,24,25,26,27].

We then apply these results to grammar modal logics [5]. We show that, for a given regular grammar Π, starting from the class of all frames, one can apply a sequence of operations of the above kind and arrive at the class of all Π-frames, i.e., frames that satisfy the set of inclusions corresponding to the rules in Π. This yields a simple proof of the result that every *regular grammar logic* (with converse) admits filtration (and hence has the FMP and is decidable). Note that the FMP for these logics was already known from [5,6].

Finally, we give two examples of logics that do not admit filtrations. To this end, we show that the global satisfiability problem is undecidable for the logics $\mathbf{K}.2 = \mathbf{K}+\Diamond\Box p \to \Box\Diamond p$ and $\mathbf{K}_2+[a]p \to [b][a][b]p$. The latter corresponds to the *irregular* grammar $a \to bab$. This enables us to build a simple undecidable context-free grammar logic, which complements Demri's paper [5].

Section 1 introduces the notion of a logic (or a class of frames) that admits filtration. In Section 2, we identify some filtration safe operations on frames and obtain the corresponding transfer results for logics (Section 2.4). In Section 3, we recall the notion of a grammar logic and give a short proof (using the results of Section 2) of the fact that every regular grammar logic (with converse) admits filtration. Section 4 presents explicit filtration constructions for logics that correspond to some subfamilies of regular grammars. Finally, Section 5 contains the above mentioned undecidability results. The paper concludes with the discussion of open questions and further directions of research.

1 Preliminaries

We assume the reader to be familiar with syntax and semantics of multi-modal logic [2,4], so we only briefly recall some notions and fix notation. Let Σ be a finite alphabet (of indices for modalities). The set $\mathsf{Fm}(\Sigma)$ of *modal formulas over* Σ is defined from propositional letters $\mathsf{Var} = \{p_0, p_1, \dots\}$ using Boolean connectives and the modalities $[e]$, for $e \in \Sigma$, according to the syntax:

$$\varphi \ ::= \ \bot \ \mid \ p_i \ \mid \ \varphi \to \psi \ \mid \ [e]\varphi.$$

We use standard abbreviations (e.g., $\top, \wedge$); in particular, $\langle e\rangle\varphi := \neg[e]\neg\varphi$. For a set of formulas Γ, by $\mathsf{Sub}(\Gamma)$ we denote the set of all subformulas of formulas from Γ. We say that Γ is Sub-*closed* if $\mathsf{Sub}(\Gamma) \subseteq \Gamma$.

A (Σ-)*frame* is a pair $F = (W, (R_e)_{e\in\Sigma})$, where $W \neq \varnothing$ and $R_e \subseteq W\times W$ for $e \in \Sigma$. A *model* based on F is a pair $M = (F, V)$, where $V(p) \subseteq W$, for all $p \in \mathsf{Var}$. The *truth relation* $M, x \models \varphi$ is defined in the usual way, e.g.

$$M, x \models [e]\varphi \quad \leftrightharpoons \quad \text{for all } y \in W, \ \text{ if } x\,R_e\,y \ \text{ then } \ M, y \models \varphi.$$

A formula φ is *valid* in F, notation $F \models \varphi$, if $M, x \models \varphi$ for all M based on F and all worlds x in F. For a class of frames $\mathcal{F}$, an $\mathcal{F}$-*model* is a model based on a frame from $\mathcal{F}$. A formula φ is *satisfiable* in $\mathcal{F}$ if it is true in some world of some $\mathcal{F}$-model; φ is *globally satisfiable* in $\mathcal{F}$ if it is true in some $\mathcal{F}$-model.

A (*modal*) *logic* (*over* Σ) is a set of formulas $\mathbf{L}$ that contains all classical tautologies, the axioms $[e](p \to q) \to ([e]p \to [e]q)$, for each $e \in \Sigma$, and is closed under the rules of modus ponens, substitution, and necessitation (from φ, infer $[e]\varphi$, for each $e \in \Sigma$). An $\mathbf{L}$-*frame* is a frame in which $\mathbf{L}$ is valid. *The logic of a class of frames* $\mathcal{F}$ is the set of all formulas that are valid in $\mathcal{F}$. A logic is *Kripke complete* if it is the logic of some class of frames. A logic $\mathbf{L}$ has the *finite model property* (FMP) if it is the logic of some class of finite frames; or equivalently (see [2, Th. 3.28]) if, for every formula $\varphi \notin \mathbf{L}$, there is a finite $\mathbf{L}$-frame F such that $F \not\models \varphi$. If, additionally, the size of F is at most exponential in size of φ, we say that $\mathbf{L}$ has the *exponential model property* (ExpMP).

1.1 Filtration

The notion of a filtration we introduce below slightly generalizes the standard one (cf. [2, Def. 2.36], [4, Sect. 5.3]) in the following aspect: given a finite set of formulas Γ, we define a filtration as a model obtained by factorizing a given model w.r.t. an equivalence relation that we allow to be *finer* than the one induced by Γ. This modification seems to first appear in [19]; see also [20].

Let $M = (W, (R_e)_{e\in\Sigma}, V)$ be a model and Γ a finite Sub-closed set of Σ-formulas. An equivalence relation $\sim$ on W is *of finite index* if the quotient set $W/{\sim}$ is finite. The equivalence relation *induced by* Γ is defined as follows:

$$x \sim_\Gamma y \quad \leftrightharpoons \quad \forall\varphi \in \Gamma \ \big(M, x \models \varphi \ \Leftrightarrow\ M, y \models \varphi\big).$$

Clearly, $\sim_\Gamma$ is of finite index. We say that an equivalence relation $\sim$ *respects* Γ if $\sim\ \subseteq\ \sim_\Gamma$; in other words, if for every $\sim$-class $\alpha \subseteq W$ and every formula $\varphi \in \Gamma$, φ is either true in all worlds of α or false in all worlds of α.

Definition 1.1 (Filtration) A *filtration* of a model M that *respects* a set of formulas Γ (or a Γ-*filtration of* M) is any model $\widehat{M} = (\widehat{W}, (\widehat{R}_e)_{e\in\Sigma}, \widehat{V})$ satisfying the following conditions:

- $\widehat{W} = W/{\sim}$, for some equivalence relation of finite index $\sim$ on W;
- the equivalence relation $\sim$ respects Γ;
- the valuation $\widehat{V}$ is defined on the variables $p \in \Gamma$ canonically: $\widehat{x} \models p \Leftrightarrow x \models p$, for all worlds $x \in W$, where $\widehat{x}$ denotes the $\sim$-class of x;
- $R_e^{\mathsf{min}} \subseteq \widehat{R}_e \subseteq \Gamma_e$, for each $e \in \Sigma$. Here R_e^{min} is the e-th *minimal filtered relation* on $\widehat{W}$, and Γ_e is the e-th *maximal filtered relation*[4] on $\widehat{W}$ induced by the set of formulas Γ; they are defined in the usual way:

$$\begin{array}{lcl} \widehat{x}\, R_e^{\mathsf{min}}\, \widehat{y} & \leftrightharpoons & \exists x' \sim x\ \exists y' \sim y:\ \ x'\, R_e\, y', \\ \widehat{x}\ \ \Gamma_e\ \ \widehat{y} & \leftrightharpoons & \text{for every formula } [e]\varphi \in \Gamma\ \big(M, x \models [e]\varphi \ \Rightarrow\ M, y \models \varphi\big). \end{array}$$

Note that the relations R_e^{min} and Γ_e are well-defined on $\sim$-classes, and that $R_e^{\mathsf{min}} \subseteq \Gamma_e$ always holds. The condition $R_e^{\mathsf{min}} \subseteq \widehat{R}_e$ is equivalent to that $\forall x, y \in W\ (x\, R_e\, y \Rightarrow \widehat{x}\, \widehat{R}_e\, \widehat{y})$. A filtration is always a finite model. The following lemma states the main property of filtrations (cf. [2, Th. 2.39], [4, Th. 5.23]).

Lemma 1.2 (Filtration lemma) *Let Γ be a finite Sub-closed set of formulas. Suppose that $\widehat{M}$ is a Γ-filtration of a model M. Then, for all worlds $x \in W$ and all formulas $\varphi \in \Gamma$, the equivalence holds: $M, x \models \varphi \Leftrightarrow \widehat{M}, \widehat{x} \models \varphi$.*

Definition 1.3 (AF) We say that a class of frames $\mathcal{F}$ *admits filtration* if, for every finite Sub-closed set of formulas Γ and every $\mathcal{F}$-model M, there exists a $\mathcal{F}$-model that is a Γ-filtration of M. A logic $\mathbf{L}$ *admits filtration* if it is Kripke complete and the class of all $\mathbf{L}$-frames admits filtration.

[4] In literature, it is sometimes denoted by $(R_e)^{\mathsf{max}}$. In fact, however, it depends not (only) on the relation R_e, but on Γ (and M). Later we use maximal filtered relations induced by different sets of formulas, so we need a notation that allows us to distinguish between them.

Example 1.4 ([9,4]). The logics **K**, **T**, **K4**, **S4**, **B**, **S5**, **S4.1**, **K**+$\Box p \to \Box^m p$, for $m \geq 0$, the multi-modal **K** (i.e., $\mathbf{K}_n$) admit filtration. The classes of point-generated **S4.2**-frames and point-generated **S4.3**-frames admit filtration.

Theorem 1.5 (AF implies FMP) *If a logic admits filtration then it has the FMP. If additionally it is finitely axiomatizable, then it is decidable.*

Proof. By the standard argument, cf. [4, Corollary 5.26]. □

A trivial (but important) remark is that any class of *finite* frames admits filtration, since any finite model is a filtration of itself. Hence, one cannot say that **L** admits filtration if it is the logic of *some* class of frames that admits filtration (in the sense of Definition 1.3), otherwise the notions of AF and FMP would coincide. Indeed, AF implies FMP; conversely, an FMP logic is the logic of the class of all finite **L**-frames, which trivially admits filtration.

Below, we need the following lemma on the relationship between some operations on relations (or on frames) and minimal filtered relations.

Lemma 1.6 *For any equivalence relation* $\sim$ *on* W *and any relations on* W*,*

(1)	$(\mathtt{Id}(W))^{\mathsf{min}} = \mathtt{Id}(\widehat{W})$	(4)	$(R^{-1})^{\mathsf{min}} = (R^{\mathsf{min}})^{-1}$
(2)	$(W \times W)^{\mathsf{min}} = \widehat{W} \times \widehat{W}$	(5)	$(R \circ S)^{\mathsf{min}} \subseteq R^{\mathsf{min}} \circ S^{\mathsf{min}}$
(3)	$(\bigcup_{i \in I} R_i)^{\mathsf{min}} = \bigcup_{i \in I} {R_i}^{\mathsf{min}}$	(6)	$(R^+)^{\mathsf{min}} \subseteq (R^{\mathsf{min}})^+$

Here $\mathtt{Id}(W) := \{(x,x) \mid x \in W\}$*, and* R^+ *is the transitive closure of* R*.*

Proof. (1), (2), and (4) are trivial. (3) follows from that $\exists$ distributes over $\vee$. It remains to prove (5), because (3) and (5) together imply (6).

(5) If $(\widehat{x}, \widehat{y}) \in (R \circ S)^{\mathsf{min}}$ then $\exists x' \sim x\ \exists y' \sim y$ with $x'\,(R \circ S)\,y'$, hence $x'\,R\,z\,S\,y'$, for some $z \in W$. Thus $\widehat{x}\,R^{\mathsf{min}}\,\widehat{z}\,S^{\mathsf{min}}\,\widehat{y}$, so $(\widehat{x}, \widehat{y}) \in (R^{\mathsf{min}} \circ S^{\mathsf{min}})$. □

2 Transferring the 'admits filtration' property

Here we point out several operations on frames that preserve the AF property, in the sense that if a class of frames admits filtration, then so does the class of all frames transformed by this operation.

2.1 Simple operations on frames

Let $\Sigma' \subset \Sigma$. Given a frame $F = (W, (R_e)_{e \in \Sigma})$, denote by $F|_{\Sigma'} = (W, (R_e)_{e \in \Sigma'})$ its Σ'-*reduct*. For a class of frames $\mathcal{F}$, denote $\mathcal{F}|_{\Sigma'} = \{F|_{\Sigma'} \mid F \in \mathcal{F}\}$. The reader can easily prove the following lemma.

Lemma 2.1 *If* $\mathcal{F}$ *admits filtration, then so does* $\mathcal{F}|_{\Sigma'}$*, for any* $\Sigma' \subset \Sigma$*.*

For a class $\mathcal{F}$ of frames of the form [5] $F = (W, R)$, let $\mathcal{F}^u := \{F^u \mid F \in \mathcal{F}\}$, where $F^u = (W, R, W \times W)$ is the frame F enriched by the *universal relation.*

Lemma 2.2 ([11, Th. 5.9]) *If* $\mathcal{F}$ *admits filtration then so does* $\mathcal{F}^u$*.*

[5] Althouth we often consider the uni-modal case below, this is for simplicity only; one can always assume that frames have additional relations, as they do not influence the proof much.

Although the notion of a logic that admits filtration used in [11] differs from ours, the same proof works for our case: simply substitute $\top$ or $\bot$ for all formulas of the form $[*]\varphi$, where $[*]$ is the *universal modality*, depending on whether φ is true or false in the given model, and then filtrate the $\mathcal{F}$-model.

Lemma 2.3 *If a class $\mathcal{F}$ of Σ-frames admits filtration, then so do the following classes of frames, for any fixed $a, b \in \Sigma$:*

(i) $\mathcal{F}^{\cup} = \{ (W, (R_e)_{e\in\Sigma}, R_a \cup R_b) \mid (W, (R_e)_{e\in\Sigma}) \in \mathcal{F} \}$,
(ii) $\mathcal{F}^{\circ} = \{ (W, (R_e)_{e\in\Sigma}, R_a \circ R_b) \mid (W, (R_e)_{e\in\Sigma}) \in \mathcal{F} \}$,
(iii) $\mathcal{F}^{=} = \{ (W, (R_e)_{e\in\Sigma}, \mathtt{Id}(W)) \mid (W, (R_e)_{e\in\Sigma}) \in \mathcal{F} \}$.

Proof. (i) To prove that $\mathcal{F}^{\cup}$ admits filtration, given a finite Sub-closed set of formulas $\Gamma \subset \mathsf{Fm}(\Sigma \cup \{c\})$, where $c \notin \Sigma$, and an $\mathcal{F}^{\cup}$-model $M^{\cup} = (W, (R_e)_{e\in\Sigma}, R_c, V)$, with $R_c = R_a \cup R_b$ and $M = (W, (R_e)_{e\in\Sigma}, V)$ an $\mathcal{F}$-model, let us show how to build an $\mathcal{F}^{\cup}$-model that is a Γ-filtration of $M^{\cup}$.

Let us introduce a translation $(\cdot)^*$ from $\mathsf{Fm}(\Sigma\cup\{c\})$ to $\mathsf{Fm}(\Sigma)$ that preserves variables, $\to$, $\bot$, $[e]$ for each $e \in \Sigma$, and satisfies: $([c]\varphi)^* = [a]\varphi^* \wedge [b]\varphi^*$. One can easily show that, for any $x \in W$ and any formula $\varphi \in \mathsf{Fm}(\Sigma \cup \{c\})$,

$$M^{\cup}, x \models \varphi \iff M, x \models \varphi^*. \qquad (*)$$

Consider the set of formulas $\Phi = \mathsf{Sub}\,\Gamma^* = \mathsf{Sub}\{\varphi^* \mid \varphi \in \Gamma\} \subseteq \mathsf{Fm}(\Sigma)$. Since $\mathcal{F}$ admits filtration, there is an $\mathcal{F}$-model $\widehat{M} = (\widehat{W}, (\widehat{R}_e)_{e\in\Sigma}, \widehat{V})$ that is a Φ-filtration of M. Here $\widehat{W} = W/{\sim}$, where $\sim$ respects Φ, $R_e^{\mathsf{min}} \subseteq \widehat{R}_e \subseteq \Phi_e$ for each $e \in \Sigma$, and $\widehat{x} \models p \Leftrightarrow x \models p$, for every variable p from $\mathsf{Var}(\Phi) = \mathsf{Var}(\Gamma)$.

Now extend $\widehat{M}$ to $N = (\widehat{W}, (\widehat{R}_e)_{e\in\Sigma}, \widehat{R}_c, \widehat{V})$ by putting $\widehat{R}_c := \widehat{R}_a \cup \widehat{R}_b$. Clearly, N is an $\mathcal{F}^{\cup}$-model. Let us show that N is a Γ-filtration of $M^{\cup}$. From $(*)$ it easily follows that $\sim$ respects Γ. It remains to prove the inclusions:

$$R_e^{\mathsf{min}} \overset{(1)}{\subseteq} \widehat{R}_e \overset{(2)}{\subseteq} \Gamma_e \quad \text{for all } e \in \Sigma, \qquad R_c^{\mathsf{min}} \overset{(3)}{\subseteq} \widehat{R}_c \overset{(4)}{\subseteq} \Gamma_c.$$

Here we already have (1); the inclusion $(R_a \cup R_b)^{\mathsf{min}} \subseteq R_a^{\mathsf{min}} \cup R_b^{\mathsf{min}}$ from Lemma 1.6(3) implies (3); (2) follows from (a) and (4) follow from (b) below.

(a) $\Phi_e \subseteq \Gamma_e$ for each $e \in \Sigma$. (Therefore, $\widehat{R}_e \subseteq \Phi_e \subseteq \Gamma_e$.)
Assume $\widehat{x}\,\Phi_e\,\widehat{y}$. To show $\widehat{x}\,\Gamma_e\,\widehat{y}$, take any $[e]\varphi \in \Gamma$. Then $[e]\varphi^* \in \Phi$ and so

$$x \models [e]\varphi \iff x \models [e]\varphi^* \implies y \models \varphi^* \iff y \models \varphi.$$

(b) $\Phi_a \cup \Phi_b \subseteq \Gamma_c$. (Here we use that $\mathcal{F}^{\cup} \models [c]p \to [a]p \wedge [b]p$.)
Assume $\widehat{x}\,(\Phi_a \cup \Phi_b)\,\widehat{y}$. Without loss of generality, $\widehat{x}\,\Phi_a\,\widehat{y}$. To prove that $\widehat{x}\,\Gamma_c\,\widehat{y}$, take any $[c]\varphi \in \Gamma$. Then $([c]\varphi)^* \in \Phi$ and so $[a]\varphi^* \in \Phi$, hence:

$$x \models [c]\varphi \iff x \models [c]\varphi^* \implies x \models [a]\varphi^* \implies y \models \varphi^* \iff y \models \varphi.$$

(ii) Use $([c]\varphi)^* = [a][b]\varphi^*$. Now (b) is: $\Phi_a \circ \Phi_b \subseteq \Gamma_c$ by Lemma 1.6(5).
(iii) Use $([c]\varphi)^* = \varphi^*$. Now $\widehat{R}_c := \mathtt{Id}(\widehat{W}) = R_c^{\mathsf{min}} \subseteq \Gamma_c$ by Lemma 1.6(1). □

2.2 Inverse relation

Given a class $\mathcal{F}$ of frames of the form $F = (W, R)$, denote $\mathcal{F}^t = \{F^t \mid F \in \mathcal{F}\}$, where $F^t = (W, R, R^{-1})$ is called the *tense expansion* of the frame F.

Theorem 2.4 *If $\mathcal{F}$ admits filtration then so does $\mathcal{F}^t$.*

Proof. Given a finite Sub-closed set of formulas $\Gamma \subset \mathsf{Fm}(\Box, \boxminus)$ and an $\mathcal{F}^t$-model $M^t = (W, R, R^{-1}, V)$, where $\Box$ refers to R and $\boxminus$ to R^{-1}, with $M = (W, R, V)$ an $\mathcal{F}$-model, we build an $\mathcal{F}^t$-model that is a Γ-filtration of M^t.

Let us introduce fresh variables $\{q_\varphi \mid \varphi \in \Gamma\}$ and extend the valuation V to them by putting:[6] $x \models q_\varphi \leftrightharpoons x \models \varphi$. Thus, $\varphi \leftrightarrow q_\varphi$ and hence $\Box\varphi \leftrightarrow \Box q_\varphi$ and $\Box\neg\varphi \leftrightarrow \Box\neg q_\varphi$ are true in M^t, for all $\varphi \in \Gamma$. Now consider the set of formulas:

$$\Phi \;:=\; \mathsf{Sub}\{\, \Box q_\varphi, \Box\neg q_\varphi \;\mid\; \varphi \in \Gamma \,\} \quad \subset \quad \mathsf{Fm}(\Box).$$

Since the class $\mathcal{F}$ admits filtration, there exists an $\mathcal{F}$-model $\widehat{M} = (\widehat{W}, \widehat{R}, \widehat{V})$ that is a Φ-filtration of M. Here $\widehat{W} = W/{\sim}$, where $\sim$ respects Φ, $R^{\mathsf{min}} \subseteq \widehat{R} \subseteq \Phi_\Box$, and $\widehat{x} \models q \Leftrightarrow x \models q$, for all variables q from Φ. Let us extend $\widehat{V}$ to the variables p from Γ by putting $\widehat{x} \models p \leftrightharpoons \widehat{x} \models q_p$.

We claim that $\widehat{M}^t := (\widehat{W}, \widehat{R}, (\widehat{R})^{-1}, \widehat{V})$ is an $\mathcal{F}^t$-model (this is obvious) and a Γ-filtration of M^t, i.e., that $\sim$ respects Γ and the inclusions $R^{\mathsf{min}} \subseteq \widehat{R} \subseteq \Gamma_\Box$ and $(R^{-1})^{\mathsf{min}} \subseteq \widehat{R}^{-1} \subseteq \Gamma_\boxminus$ hold.

(a) *The equivalence relation $\sim$ respects the set of formulas Γ.*
Assume that $x \sim y$. Then, since $\sim$ respects Φ, we have the equivalences:

$$x \models \varphi \quad\Longleftrightarrow\quad x \models q_\varphi \quad\Longleftrightarrow\quad y \models q_\varphi \quad\Longleftrightarrow\quad y \models \varphi.$$

(b) $(R^{-1})^{\mathsf{min}} = (R^{\mathsf{min}})^{-1}$. (By Lemma 1.6(4).)

(c) $\Phi_\Box \subseteq \Gamma_\Box$.
Assume $\widehat{x}\,\Phi_\Box\,\widehat{y}$. To show $\widehat{x}\,\Gamma_\Box\,\widehat{y}$, take any $\Box\varphi \in \Gamma$. Then $\Box q_\varphi \in \Phi$ and so

$$x \models \Box\varphi \quad\Longleftrightarrow\quad x \models \Box q_\varphi \quad\Longrightarrow\quad y \models q_\varphi \quad\Longleftrightarrow\quad y \models \varphi.$$

(d) $(\Phi_\Box)^{-1} \subseteq \Gamma_\boxminus$. (The proof of this item contains the main trick.)
Assume $\widehat{x}\,\Phi_\Box\,\widehat{y}$. To show $\widehat{y}\,\Gamma_\boxminus\,\widehat{x}$, take any $\varphi := \boxminus\alpha \in \Gamma$. Then $\Box\neg q_\varphi \in \Phi$.
We prove the required implication: $y \models \boxminus\alpha \Rightarrow x \models \alpha$, by contraposition:

$$x \not\models \alpha \;\overset{(1)}{\Longrightarrow}\; x \models \Box\neg\boxminus\alpha \;\overset{(2)}{\Longleftrightarrow}\; x \models \Box\neg q_\varphi \;\overset{(3)}{\Longrightarrow}\; y \models \neg q_\varphi \;\overset{(4)}{\Longleftrightarrow}\; y \not\models \boxminus\alpha$$

Here (1) follows from that $F \models \neg p \to \Box\neg\boxminus p$; (2) and (4) hold since $\varphi \leftrightarrow q_\varphi$ and hence $\boxminus\alpha \leftrightarrow q_\varphi$ are true in M^t; (3) holds since $\Box\neg q_\varphi \in \Phi$ and $\widehat{x}\,\Phi_\Box\,\widehat{y}$.

Thus, $R^{\mathsf{min}} \subseteq \widehat{R} \subseteq \Phi_\Box \subseteq \Gamma_\Box$ and $(R^{-1})^{\mathsf{min}} = (R^{\mathsf{min}})^{-1} \subseteq \widehat{R}^{-1} \subseteq (\Phi_\Box)^{-1} \subseteq \Gamma_\boxminus$. □

As a corollary, we obtain the following. Given a frame $F = (W, (R_e)_{e \in \Sigma})$, let $F^\ominus := (W, (R_e^{-1})_{e \in \Sigma})$. For a class of frames $\mathcal{F}$, put $\mathcal{F}^\ominus = \{F^\ominus \mid F \in \mathcal{F}\}$.

Theorem 2.5 (Inverting) *If $\mathcal{F}$ admits filtration, then so does $\mathcal{F}^\ominus$.*

Proof. First, add inverse relations, using Theorem 2.4. Secongly, drop the original relations, i.e., take the $\overline{\Sigma}$-reduct, using Lemma 2.1. □

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[6] Throughout the proof, we write $x \models \varphi$ instead of $M, x \models \varphi$ or $M^t, x \models \varphi$. This is unambiguous, since the truth values of $\Box$-formulas in M and M^t coincide, while for formulas involving $\boxminus$, the shortcut $x \models \varphi$ simply means $M^t, x \models \varphi$.

2.3 Transitive closure

Given a class $\mathcal{F}$ of frames of the form $F = (W, R)$, denote $\mathcal{F}^{\oplus} = \{F^{\oplus} \mid F \in \mathcal{F}\}$, where $F^{\oplus} = (W, R, R^+)$ and $R^+ = \bigcup_{n\geq 1} R^n$ is the *transitive closure* of R.

Theorem 2.6 *If $\mathcal{F}$ admits filtration then so does $\mathcal{F}^{\oplus}$.*

Proof. Given a finite Sub-closed set of formulas $\Gamma \subset \mathsf{Fm}(\Box, \boxplus)$ and an $\mathcal{F}^{\oplus}$-model $M^{\oplus} = (W, R, R^+, V)$, where $\Box$ refers to R and $\boxplus$ to R^+, with $M = (W, R, V)$ an $\mathcal{F}$-model, we build an $\mathcal{F}^{\oplus}$-model that is a Γ-filtration of $M^{\oplus}$.

Let us introduce fresh variables $\{q_\varphi \mid \varphi \in \Gamma\}$ and extend the valuation V to them by putting: $x \models q_\varphi \leftrightharpoons x \models \varphi$. Thus, $\varphi \leftrightarrow q_\varphi$ and hence $\Box\varphi \leftrightarrow \Box q_\varphi$ are true in $M^{\oplus}$. Now consider the following set of formulas:

$$\Phi \ := \ \{\, q_\varphi,\, \Box q_\varphi \ \mid \ \varphi \in \Gamma \,\} \quad \subset \quad \mathsf{Fm}(\Box).$$

Since the class $\mathcal{F}$ admits filtration, there exists an $\mathcal{F}$-model $\widehat{M} = (\widehat{W}, \widehat{R}, \widehat{V})$ that is a Φ-filtration of M. Here $\widehat{W} = W/{\sim}$, where $\sim$ respects Φ, $R^{\mathsf{min}} \subseteq \widehat{R} \subseteq \Phi_\Box$, and $\widehat{x} \models q \Leftrightarrow x \models q$, for all variables q from Φ. Let us extend $\widehat{V}$ to the variables p from Γ by putting: $\widehat{x} \models p \leftrightharpoons \widehat{x} \models q_p$.

We claim that the model $\widehat{M}^{\oplus} := (\widehat{W}, \widehat{R}, (\widehat{R})^+, \widehat{V})$ is an $\mathcal{F}^{\oplus}$-model (this is obvious) and a Γ-filtration of $M^{\oplus}$, i.e., that $\sim$ respects Γ and the inclusions $R^{\mathsf{min}} \subseteq \widehat{R} \subseteq \Gamma_\Box$ and $(R^+)^{\mathsf{min}} \subseteq \widehat{R}^+ \subseteq \Gamma_\boxplus$ hold.

(a) *The equivalence relation $\sim$ respects the set Γ.* (As in Theorem 2.4.)
(b) $(R^+)^{\mathsf{min}} \subseteq (R^{\mathsf{min}})^+$. (By Lemma 1.6(6).)
(c) $\Phi_\Box \subseteq \Gamma_\Box$. (As in Theorem 2.4.)
(d) $\Phi_\Box \subseteq \Gamma_\boxplus$. (Here we will use that $F^{\oplus} \models \boxplus p \to \Box p$.)
Assume $\widehat{x}\,\Phi_\Box\,\widehat{y}$. To prove $\widehat{x}\,\Gamma_\boxplus\,\widehat{y}$, take any $\boxplus\varphi \in \Gamma$, then $\Box q_\varphi \in \Phi$ and so:
$x \models \boxplus\varphi \ \Rightarrow\ x \models \Box\varphi \ \Leftrightarrow\ x \models \Box q_\varphi \ \Rightarrow\ y \models q_\varphi \ \Leftrightarrow\ y \models \varphi$. Thus $\widehat{x}\,\Gamma_\boxplus\,\widehat{y}$.
(e) $\Phi_\Box \circ \Gamma_\boxplus \subseteq \Gamma_\boxplus$. (Here we will use that $F^{\oplus} \models \boxplus p \to \Box\boxplus p$.)
Assume $\widehat{x}\,\Phi_\Box\,\widehat{y}\,\Gamma_\boxplus\,\widehat{z}$. To show $\widehat{x}\,\Gamma_\boxplus\,\widehat{z}$, take any $\boxplus\varphi \in \Gamma$. Then $\Box q_{\boxplus\varphi} \in \Phi$, so:
$x \models \boxplus\varphi \ \Rightarrow\ x \models \Box\boxplus\varphi \ \Leftrightarrow\ x \models \Box q_{\boxplus\varphi} \ \Rightarrow\ y \models q_{\boxplus\varphi} \ \Leftrightarrow\ y \models \boxplus\varphi \ \Rightarrow\ z \models \varphi$.
(f) $(\Phi_\Box)^+ \subseteq \Gamma_\boxplus$.
It suffices to prove $(\Phi_\Box)^n \subseteq \Gamma_\boxplus$, by induction on $n \geq 1$. Induction base is (d); induction step: $(\Phi_\Box)^{n+1} = \Phi_\Box \circ (\Phi_\Box)^n \subseteq \Phi_\Box \circ \Gamma_\boxplus \subseteq \Gamma_\boxplus$, by (e).

Thus, $R^{\mathsf{min}} \subseteq \widehat{R} \subseteq \Phi_\Box \subseteq \Gamma_\Box$ and $(R^+)^{\mathsf{min}} \subseteq (R^{\mathsf{min}})^+ \subseteq \widehat{R}^+ \subseteq (\Phi_\Box)^+ \subseteq \Gamma_\boxplus$. □

An analogue of Theorem 2.6 holds for the *reflexive-transitive closure* $R^* = \mathtt{Id}(W) \cup R^+$, where additionally one needs to use Lemmas 2.3 and 2.1.

2.4 Operations on logics

Let $\mathbf{L}$ be a logic $\mathbf{L}$ over Σ. Denote by $\mathbf{L}^u$ its extension with the universal modality $[*]$ and the axioms, for all $e \in \Sigma$ (the last three are **S5**-axioms for $[*]$):

$$[*]p \to [e]p, \qquad [*]p \to p, \qquad [*]p \to [*][*]p, \qquad \neg[*]p \to [*]\neg[*]p.$$

The *tense counterpart* $\mathbf{L}^t$ of $\mathbf{L}$ is the logic over $\Sigma \cup \overline{\Sigma}$ that extends $\mathbf{L}$ with the following *tense axioms*, for all $e \in \Sigma$:

$$p \to [e]\langle\overline{e}\rangle p, \qquad p \to [\overline{e}]\langle e\rangle p.$$

Given a logic **L** over $\Sigma = \{\Box\}$, denote by $\mathbf{L}^{\boxplus}$ the extension of **L** with the *transitive closure* modality $\boxplus$ and the following axioms:

$$\boxplus p \to \Box p, \qquad \boxplus p \to \Box\boxplus p, \qquad \boxplus(p \to \Box p) \to (\Box p \to \boxplus p).$$

It is known that extending a logic even with a seemingly "harmless" universal modality is not safe: we can lose the FMP [22], decidability [21], and even Kripke completeness [14, Corollary 9.6.5]. A number of negative results are also known for the tense extension and the transitive closure extension, see e.g. [23,24], [2, Theorem 6.34]. However, if the extended logic is Kripke complete, then we can obtain the desired transfer results for logics.

Theorem 2.7 (Filtration safe operations on logics) *Suppose that a logic* **L** *admits filtration.*

(i) *If the logic* $\mathbf{L}^u$ *is Kripke complete then it admits filtration [11].*
(ii) *If the logic* $\mathbf{L}^t$ *is Kripke complete then it admits filtration.*
(iii) *If the logic* $\mathbf{L}^{\boxplus}$ *is Kripke complete then it admits filtration.*

Proof. Let $\mathcal{F}$ be the class of all **L**-frames. As shown in [11], if $\mathbf{L}^u$ is complete then it is the logic of $\mathcal{F}^u$. Furthermore, it is known (see e.g. [14]) that $F \models \mathbf{L}$ iff $F^t \models \mathbf{L}^t$. Similarly, $F \models \mathbf{L}$ iff $F^{\oplus} \models \mathbf{L}^{\boxplus}$. Now, (i)–(iii) follow from the Kripke completeness and Lemma 2.2, Theorems 2.4 and 2.6. □

Corollary 2.8 *If a logic* **L** *admits filtration and is finitely axiomatizable, and* $\mathbf{L}^u$ *is Kripke complete, then the global satisfiability problem for* **L** *is decidable.*

Proof. Let $\mathcal{F}$ be the class of all **L**-frames. Then, for any formula φ, we have: φ is globally satisfiable in **L** iff the formula $[*]\varphi$ is satisfiable in $\mathbf{L}^u$. □

The next lemma gives a sufficient condition for the completeness of $\mathbf{L}^u$ and $\mathbf{L}^t$.

Lemma 2.9 *If a logic* **L** *is canonical then* $\mathbf{L}^t$ *and* $\mathbf{L}^u$ *are Kripke complete.*

Proof. The axioms for $[*]$ and the tense axioms are canonical formulas. □

Wolter [23,24,25,26,27] obtained a lot of general transfer results for $\mathbf{L}^t$. However, they seem not to cover our Theorem 2.7(ii), so we believe the latter is new. At the same time, we are not aware of any transfer results for $\mathbf{L}^{\boxplus}$, nor general sufficient conditions for it to be Kripke complete.

3 Grammar logics

Here we show that the above results easily imply that every regular grammar logic (with converse) admits filtration and hence has the FMP. The result on FMP is not new, as there is a translation from regular grammar logics (with converse) into **PDL** with finite automata as modalities [5] (into the guarded fragment of the first-order logic with two variables $\mathbf{GF}^2$ [6], respectively), therefore, the FMP result for regular grammar logics (with converse) follows from the FMP for **PDL** and $\mathbf{GF}^2$ obtained in [7] and [17], respectively.

3.1 Grammars

By a *grammar* [7] over an alphabet Σ we mean a finite set Π of (*production*) *rules* of the form $u \to v$, where $u, v \in \Sigma^*$, $u \neq \varepsilon$ (here ε is the empty word). Below, we only deal with *context-free* grammars, whose rules have the form $e \to v$, where $e \in \Sigma$, although some definitions are applicable to arbitrary grammars. We say that a rule $u \to v$ transforms, for any $x, y \in \Sigma^*$, the word xuy into xvy, and denote this relation on words by $xuy \overset{\Pi}{\longmapsto} xvy$. Let [8] $\overset{\Pi}{\Longmapsto}$ be the reflexive-transitive closure of the relation $\overset{\Pi}{\longmapsto}$. The set of all words producible from a given word u is denoted by $\Pi(u) = \{v \in \Sigma^* \mid u \overset{\Pi}{\Longmapsto} v\}$.

A grammar Π is called *regular* if, for every $e \in \Sigma$, the language $\Pi(e)$ is regular. Recall that *regular languages* are obtained from the empty language [9] $\varnothing$, the singleton languages $\{\varepsilon\}$ and $\{e\}$, for all $e \in \Sigma$, using the operations of union $L_1 \cup L_2$, composition $L_1 \circ L_2$ and Kleene star L^* on languages (cf. [13]).

We also consider grammars over the alphabet $\Sigma \cup \overline{\Sigma}$, where the symbol $\overline{e}$ in $\overline{\Sigma}$ is called the *inverse* of the corresponding symbol $e \in \Sigma$. In a frame, we have $R_{\overline{e}} = (R_e)^{-1}$. The *inverse* of a word $u = e_1 \dots e_n$ is defined as $\overline{u} := \overline{e}_n \dots \overline{e}_1$.

3.2 Grammar modal logics

In the modal language over Σ, we have the modality $[e]$ for each $e \in \Sigma$. For any word $u = e_1 \dots e_n$ over Σ, we denote the modal operator $[u] := [e_1] \dots [e_n]$. To a rule $u \to v$, we associate the formula $[u]p \to [v]p$. For a grammar Π, its *grammar (modal) logic* $\mathbf{K}\Pi$ is the extension of the minimal normal multi-modal Σ-logic $\mathbf{K}_\Sigma$, or its tense extension $\mathbf{K}^t_\Sigma$ (see Section 2.4) in case we have a grammar over $\Sigma \cup \overline{\Sigma}$, with the axioms $[u]p \to [v]p$, for all rules $(u \to v) \in \Pi$.

In a Σ-frame $F = (W, (R_e)_{e \in \Sigma})$, any word $u = e_1 \dots e_n$ in Σ^* gives rise to the relation $R_u := R_{e_1} \circ \dots \circ R_{e_n}$. It is easily seen that $F \models [u]p \to [v]p$ iff $R_u \supseteq R_v$ (notice the converse inclusion!); in this case we write $F \models (u \to v)$. We say that F is a Π-*frame* and write $F \models \Pi$ if $F \models (u \to v)$, for all rules $(u \to v) \in \Pi$. A Π-*model* is a model based on a Π-frame. Since $R_{\overline{u}} = (R_u)^{-1}$, we have $F \models u \to v$ iff $F \models \overline{u} \to \overline{v}$, for any frame F over $\Sigma \cup \overline{\Sigma}$.

For any grammar Π, even over $\Sigma \cup \overline{\Sigma}$, the modal logic $\mathbf{K}\Pi$ is Kripke complete w.r.t. the class of Π-frames, since axioms of the form $[u]p \to [v]p$ are Sahlqvist formulas, see [2, Th. 4.42]. One of the main problems in this field is to determine for which grammars Π the logic $\mathbf{K}\Pi$ is decidable, or even has the FMP. Below, we show that, for every *regular* grammar Π, the logic $\mathbf{K}\Pi$ admits filtration and hence has the FMP and is decidable.

3.3 Regular grammar logics admit filtration

Let Π be a (context-free) grammar over Σ and $F = (W, (R_e)_{e \in \Sigma})$ a Σ-frame. The Π-*closure* of F is the frame $F^\Pi := (W, (R^\Pi_e)_{e \in \Sigma})$, where $R^\Pi_e = \bigcup_{v \in \Pi(e)} R_v$. Since $e \overset{\Pi}{\Longmapsto} e$, we have $R_e \subseteq R^\Pi_e$.

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[7] Our definition does not include a start symbol, in which aspect it is closer to a *semi-Thue system*; however, we will need a distinction between terminal and non-terminal symbols.

[8] In texts on formal language theory, this relation is often denoted by $u \Rightarrow^*_\Pi v$.

[9] In fact, we do not need $\varnothing$ below, since we always have $\Pi(e) \neq \varnothing$ due to that $e \overset{\Pi}{\Longmapsto} e$.

Lemma 3.1 (a) *The* Π*-closure of any frame* F *is a* Π*-frame:* $F^\Pi \models \Pi$.
(b) *If* F *is a* Π*-frame then* $F^\Pi = F$.

Proof. (a) For each $(e \to u) \in \Pi$, let us show that $R_e^\Pi \supseteq R_u^\Pi$. Let $u = e_1 \dots e_n$. If $x\, R_u^\Pi\, y$, i.e., $x\,(R_{e_1}^\Pi \circ \dots \circ R_{e_n}^\Pi)\, y$, then $x\,(R_{v_1} \circ \dots \circ R_{v_n})\, y$, for some $v_i \in \Pi(e_i)$, hence $x\,(R_{v_1 \dots v_n})\, y$. Since $e \overset{\Pi}{\longmapsto} e_1 \dots e_n$ and $e_i \overset{\Pi}{\Longmapsto} v_i$, we have $e \overset{\Pi}{\Longmapsto} v_1 \dots v_n$. Thus, $x\, R_v\, y$ for the word $v := v_1 \dots v_n \in \Pi(e)$. Therefore, $x\, R_e^\Pi\, y$.

(b) Assume $F \models \Pi$. Let us show that $F^\Pi = F$, i.e., $R_e^\Pi = R_e$. Here '$\supseteq$' is trivial. To prove '$\subseteq$', note that $R_u \subseteq R_a$, for all rules $(a \to u) \in \Pi$. Then $R_v \subseteq R_a$, for all $v \in \Pi(a)$, by induction on derivation in Π. Thus, $R_e^\Pi \subseteq R_e$. □

By the above lemma, taking the Π-closure of all frames yields exactly the class of *all* Π-frames. Next, we show that in case Π is a regular grammar, the Π-closure can be obtained by finitely many operations $\cup, \circ, *$.

Let $\Sigma = \{e_1, \dots, e_n\}$ and let $E = E(e_1, \dots, e_n)$ be a regular expression over Σ, i.e., it is built up from ε and e_i using $\cup, \circ, *$. Denote by $\mathbb{L}(E)$ the (regular) language represented by E, see [13]. In a frame $F = (W, R_{e_1}, \dots, R_{e_n})$, we can "substitute" $\mathtt{Id}(W)$ for ε and R_{e_i} for e_i into the expression E, thus obtaining a relation $E(R_{e_1}, \dots, R_{e_n})$ built up from $\mathtt{Id}(W)$ and R_{e_i} using $\cup, \circ, *$.

Lemma 3.2 $E(R_{e_1}, \dots, R_{e_n}) = \bigcup_{v \in \mathbb{L}(E)} R_v$*, for any regular expression* E.

Proof. By an easy induction on the complexity of the expression E. □

Now, if Π is regular, then each language $\Pi(e)$ is represented by some regular expression E_e, i.e., $\Pi(e) = \mathbb{L}(E_e)$. Then $R_e^\Pi = E_e(R_{e_1}, \dots, R_{e_n})$ by Lemma 3.2. So, $\mathcal{F}^\Pi$ is obtained from $\mathcal{F}$ by applying the frame operations $\cup, \circ, *$ (see Sections 2.1 and 2.3) finitely many times. Thus we obtain the following result.

Theorem 3.3 *For every regular grammar* Π *over* $\Sigma \cup \overline{\Sigma}$*, the class of* Π*-frames admits filtration. Consequently, every regular grammar logic (with converse) admits filtration, has the FMP, and is decidable; moreover, it has the ExpMP.*

Proof. Starting from the class of all Σ-frames, first add the inverse relations $R_{\overline{e}}$, for each $e \in \Sigma$, by applying the tense expansion operation from Section 2.2. Then apply the operations $\cup, \circ, *$ on frames from Sections 2.1 and 2.3 to obtain the regular expressions E_e that represent the languages $\Pi(e)$, for each $e \in \Sigma$. Finally, in the resulting frames, drop the initial relations R_e and all the relations built at intermediate steps, using Lemma 2.1, and arrange the final relations R_e^Π in the same order as the initial relations R_e. Thus we obtain the class of all Π-frames. Since the class of all Σ-frames admits filtration (see Example 1.4) and the above operations are filtration safe, it follows that the class of all Π-frames admits filtration as well.

Moreover, at each step, the set of formulas grows linearly: $|\Phi| \leq 3|\Gamma|$ (see Lemma 2.3, Theorems 2.4 and 2.6). Hence, starting with a filtration for the class of all $(\Sigma \cup \overline{\Sigma})$-frames through the set $\mathsf{Sub}(\varphi)$ of size $O(|\varphi|)$, after a fixed number of operations, we arrive at a filtration through a set of formulas of size $O(|\varphi|)$. Thus we obtain a countermodel of the size exponential in $|\varphi|$. □

4 Filtration for some regular grammar logics

Here we present filtration constructions for some familiar classes of regular grammar logics (with converse). First, for *right-linear* grammar logics, we give two different constructions of filtration. Then we adjust them to cover the converse modalities for terminal symbols. Next, we adapt the construction to *left-linear* grammar logics. Finally, we consider grammar logics with *left-* and *right-recursive* rules.

4.1 On maximal filtered relations

In Definition 1.1, we introduced the minimal R_e^{min} and the (Γ-)maximal Γ_e filtered relations on $\widehat{W}$, for every symbol $e \in \Sigma$. Consequently, for any word $u = e_1 \dots e_n \in \Sigma^*$, the relations $R_u^{\mathsf{min}} = R_{e_1}^{\mathsf{min}} \circ \dots \circ R_{e_n}^{\mathsf{min}}$ and $\Gamma_u = \Gamma_{e_1} \circ \dots \circ \Gamma_{e_n}$ are well-defined. Let us also introduce the maximal filtered relation $\Gamma_{[u]}$ on $\widehat{W}$ induced by the set Γ and the "compound" modality $[u] = [e_1] \dots [e_n]$:

$$\widehat{x}\ \Gamma_{[u]}\ \widehat{y} \quad \leftrightharpoons \quad \text{for every formula } [u]\varphi \in \Gamma\ \big(x \models [u]\varphi\ \Rightarrow\ y \models \varphi\big).$$

Lemma 4.1 $\Gamma_{[u]} \circ \Gamma_{[v]} \subseteq \Gamma_{[uv]}$, *for all words* $u, v \in \Sigma^*$.

Proof. Assume $\widehat{x}\,\Gamma_{[u]}\,\widehat{y}\,\Gamma_{[v]}\,\widehat{z}$. To prove $\widehat{x}\,\Gamma_{[uv]}\,\widehat{z}$, take any $[uv]\varphi \in \Gamma$. Since Γ is Sub-closed, we have $[v]\varphi \in \Gamma$ and $\varphi \in \Gamma$. Therefore, $x \models [u][v]\varphi$ implies $y \models [v]\varphi$, which in turn implies $z \models \varphi$, as required. □

If we deal with the modal language over $\Sigma \cup \overline{\Sigma}$, the following relations are useful: $\Gamma_u^\sharp := (\Gamma_{\overline{u}})^{-1}$ and $\Gamma_{[u]}^\sharp := (\Gamma_{[\overline{u}]})^{-1}$. Explicitly, for $e \in \Sigma$, we define:

$$\widehat{x}\,\Gamma_e^\sharp\,\widehat{y} \quad \leftrightharpoons \quad \text{for every formula } [\overline{e}]\varphi \in \Gamma\ \big(y \models [\overline{e}]\varphi\ \Rightarrow\ x \models \varphi\big).$$

Lemma 4.2 (Minimax) *For any words* $u, v \in (\Sigma \cup \overline{\Sigma})^*$, *we have:*

- **(a)** $R_u^{\mathsf{min}} \subseteq \Gamma_u \subseteq \Gamma_{[u]}$,
- **(b)** $R_u^{\mathsf{min}} \subseteq \Gamma_u^\sharp \subseteq \Gamma_{[u]}^\sharp$.

Proof. **(a)** The first inclusion, $R_u^{\mathsf{min}} \subseteq \Gamma_u$, follows from the trivial inclusion $R_{e_i}^{\mathsf{min}} \subseteq \Gamma_{e_i}$ for all i, by monotonicity of the composition. As for the second inclusion, we have $\Gamma_u = \Gamma_{e_1} \circ \dots \circ \Gamma_{e_n} = \Gamma_{[e_1]} \circ \dots \circ \Gamma_{[e_n]} \subseteq \Gamma_{[u]}$, where we used the trivial equality $\Gamma_{e_i} = \Gamma_{[e_i]}$ and Lemma 4.1.

(b) This follows from **(a)**, once we observe that $(R_u^{\mathsf{min}})^{-1} = R_{\overline{u}}^{\mathsf{min}}$. □

4.2 Filtration for right-linear grammar logics

Let Π be a *right-linear grammar* over[10] $\Sigma = T \cup N$, which means that Π consists of rules of the form $a \to uc$, where $a, c \in N$ and $u \in T^*$. Given a model $M = (F, V)$ based on a Π-frame $F = (W, (R_e)_{e\in\Sigma})$ and a finite Sub-closed set of formulas $\Gamma \subseteq \mathsf{Fm}(\Sigma)$, we will build a Π-model $\widehat{M}$ that is a Γ-filtration of M.

We introduce the following operators on (finite Sub-closed) sets of formulas:

[10] We partition the set of symbols Σ into two disjoint sets of the so-called *terminal* and *non-terminal* symbols. In any rule $a \to u$ of any grammar, we always assume that $a \in N$.

$$N(\Psi) \;:=\; \mathsf{Sub}\big\{\,[c]\varphi \mid a,c\in N,\ [a]\varphi\in\Psi\,\big\},$$
$$\Pi(\Psi) \;:=\; \mathsf{Sub}\big\{\,[v]\varphi \mid [a]\varphi\in\Psi,\ (a\to v)\in\Pi\,\big\}.$$

Now put $\Delta := \Gamma \cup N(\Gamma)$ and $\Phi := \Delta \cup \Pi(\Delta)$. We filter the model M *through* Φ, i.e., we consider the equivalence relation $\sim_\Phi$ (see Section 1.1), which obviously has the finite index and respects the set Γ, and put $\widehat{W} := W/{\sim_\Phi}$.

Lemma 4.3 (Max-frame) $\Delta_a \supseteq \Phi_u \circ \Delta_c$, *for every rule* $(a \to uc) \in \Pi$.

Proof. Assume $\widehat{x}\,\Phi_u\,\widehat{y}\,\Delta_c\,\widehat{z}$. To show that $\widehat{x}\,\Delta_a\,\widehat{z}$, take any $[a]\varphi\in\Delta$. Then:

$$x \models [a]\varphi \quad\overset{(1)}{\Longrightarrow}\quad x\models [u][c]\varphi \quad\overset{(2)}{\Longrightarrow}\quad y\models [c]\varphi \quad\overset{(3)}{\Longrightarrow}\quad z\models\varphi.$$

Here **(1)** is due to that $F\models (a\to uc)$, so that $M,x\models [a]\varphi\to[u][c]\varphi$; **(2)** holds since $[u][c]\varphi\in\Pi(\Delta)\subseteq\Phi$ and $(\widehat{x},\widehat{y})\in\Phi_u\subseteq\Phi_{[u]}$ by the Minimax Lemma; finally, **(3)** holds since $\widehat{y}\,\Delta_c\,\widehat{z}$ and $[c]\varphi\in N(\Delta)\subseteq\Delta$. □

Next, we define the valuation $\widehat{V}$ on the variables p from $\mathsf{Var}(\Phi)=\mathsf{Var}(\Gamma)$ canonically: $\widehat{x}\models p$ iff $x\models p$. In order to obtain the frame $\widehat{F}=(\widehat{W},(\widehat{R}_e)_{e\in\Sigma})$ and the model $\widehat{M}=(\widehat{F},\widehat{V})$, it remains to define the relations $\widehat{R}_e$ so that $\widehat{F}\models\Pi$ and $\widehat{R}_e\subseteq\Gamma_e$, for each $e\in\Sigma$. We give two different constructions for this.

4.2.1 Right-linear grammars: Mini-maximal frame

Let $\widehat{R}_e$ be the minimal (for $e\in T$) or the Δ-maximal (for $e\in N$) relation:

$$\widehat{R}_e \;:=\; \begin{cases} R_e^{\mathsf{min}}, & \text{if } e\in T;\\ \Delta_e, & \text{if } e\in N.\end{cases}$$

Lemma 4.4 $\widehat{F}\models\Pi$.

Proof. Take any rule $(a\to uc)\in\Pi$, where $a,c\in N$ and $u\in T^*$. Then the required inclusion $\widehat{R}_a\supseteq\widehat{R}_u\circ\widehat{R}_c$ follows from Lemma 4.3, since $\widehat{R}_a=\Delta_a$, $\widehat{R}_c=\Delta_c$, and $\widehat{R}_u=R_u^{\mathsf{min}}\subseteq\Phi_u$, for any $u\in T^*$, by the Minimax Lemma. □

Finally, $\widehat{R}_e\subseteq\Gamma_e$, for each $e\in\Sigma$. Indeed, for $e\in T$ this is obvious, and for $e\in N$, we have that $\widehat{R}_e=\Delta_e\subseteq\Gamma_e$, because $\Gamma\subseteq\Delta$.

Remark 4.5 The above proof remains valid if, for $e\in T$, we put $\widehat{R}_e:=\Phi_e$ (or even any relation between R_e^{min} and Φ_e). However, this does not generalize to the logic with converse terminals, while the relations R_e^{min} still work for them.

4.2.2 Right-linear grammars: Π-closure

This time, we put $\widehat{R}_e:=R_e^{\mathsf{min}}$ for $e\in T$, while for $a\in N$, we define $\widehat{R}_a$ using the Π-*closure*. That is, we define, simultaneously for all $a\in N$, a tower of relations $R_a^{(0)}\subseteq R_a^{(1)}\subseteq R_a^{(2)}\subseteq\ldots$ by induction:

$$R_a^{(0)}:=R_a^{\mathsf{min}},\quad R_a^{(n+1)}:=R_a^{(n)}\;\cup\bigcup_{(a\to uc)\in\Pi}\big(\widehat{R}_u\circ R_c^{(n)}\big),\quad \widehat{R}_a:=\textstyle\bigcup_n R_a^{(n)}.\ (*)$$

Note that in $(*)$ we have $u\in T^*$, so that $\widehat{R}_u$ is already defined via $\widehat{R}_e$, $e\in T$.

Lemma 4.6 $\widehat{F}\models\Pi$.

Proof. Since $\widehat{W}$ is finite, we can find the stage n at which the sequence in $(*)$ stabilizes for all $a \in N$. Now, for any rule $(a \to uc) \in \Pi$, we have $\widehat{R}_c = R_c^{(n)}$ and $\widehat{R}_a = R_a^{(n+1)}$. Then $(*)$ implies the required inclusion $\widehat{R}_a \supseteq \widehat{R}_u \circ \widehat{R}_c$. $\square$

Lemma 4.7 $\widehat{R}_a \subseteq \Delta_a$, *for every non-terminal* $a \in N$.

Proof. It suffices to prove, by induction on n, that, for all $a \in N$, $R_a^{(n)} \subseteq \Delta_a$. Induction base is trivial: $R_a^{\mathsf{min}} \subseteq \Delta_a$. Induction step: in the expression $(*)$ for $R_a^{(n+1)}$, all terms are contained in Δ_a. Indeed, $R_a^{(n)} \subseteq \Delta_a$ by I.H., and for every rule $(a \to uc) \in \Pi$, by the Max-frame Lemma,

$$\widehat{R}_u \circ R_c^{(n)} \subseteq \Phi_u \circ \Delta_c \subseteq \Delta_a,$$

where $\widehat{R}_u \subseteq \Phi_u$, by the Minimax Lemma, and $R_c^{(n)} \subseteq \Delta_c$, by I.H. for $c \in N$. $\square$

Thus, $\widehat{R}_e \subseteq \Gamma_e$ for $e \in T$; and for $e \in N$, we have $\widehat{R}_e \subseteq \Delta_e \subseteq \Gamma_e$, as $\Gamma \subseteq \Delta$.

4.3 Right-linear grammar logics with converse terminals

Let Π be a right-linear grammar with converse terminals, i.e., its rules have the form $a \to uc$, where $a, c \in N$ and $u \in (T \cup \overline{T})^*$. We adjust the "mini-maximal frame" construction. (We could adjust the "Π-closure" construction as well.)

In addition to the operators $N(\Psi)$ and $\Pi(\Psi)$ defined above, we introduce

$$S(\Psi) \quad := \quad \mathsf{Sub}\left\{ [a]\neg[\overline{a}]\varphi \;\mid\; [\overline{a}]\varphi \in \Psi \right\}.$$

Now, given a set Γ, we put $\Lambda = \Gamma \cup S(\Gamma)$, $\Delta = \Lambda \cup N(\Lambda)$, and $\Phi = \Delta \cup \Pi(\Delta)$.

Then the proof proceeds as in Section 4.2.1, so we have: (a) $R_e^{\mathsf{min}} \subseteq \widehat{R}_e$, and (b) $\widehat{R}_e \subseteq \Gamma_e$, for all $e \in \Sigma$. However, in presence converse modalities, we also need to prove: (c) $R_{\overline{e}}^{\mathsf{min}} \subseteq \widehat{R}_{\overline{e}}$, and (d) $\widehat{R}_{\overline{e}} \subseteq \Gamma_{\overline{e}}$, for all $e \in \Sigma$. Here (c) follows trivially from (a); while (d) for $e \in T$ is trivial, since $\widehat{R}_e = R_e^{\mathsf{min}}$. The next lemma proves (d) for $e \in N$; its proof resembles the one for Theorem 2.4.

Lemma 4.8 $\Delta_a \subseteq \Gamma_a^\sharp$, *for every non-terminal* $a \in N$.

Proof. Assume $\widehat{x}\,\Delta_a\,\widehat{y}$. To prove that $\widehat{x}\,\Gamma_a^\sharp\,\widehat{y}$, we take any formula $[\overline{a}]\varphi \in \Gamma$ and show that $y \models [\overline{a}]\varphi$ implies $x \models \varphi$. The proof is by contraposition:

$$x \not\models \varphi \quad \overset{(1)}{\Longrightarrow} \quad x \models [a]\neg[\overline{a}]\varphi \quad \overset{(2)}{\Longrightarrow} \quad y \not\models \neg[\overline{a}]\varphi.$$

Here **(1)** is due to that $F \models \neg p \to [a]\neg[\overline{a}]p$, while **(2)** follows from that $\widehat{x}\,\Delta_a\,\widehat{y}$ and $[a]\neg[\overline{a}]\varphi \in S(\Gamma) \subseteq \Delta$. $\square$

4.4 Left-linear grammars (with converse terminals)

For left-linear grammars, the claim follows from the one for the corresponding right-linear grammars, due to Theorem 2.5. Indeed, inverting all relations in frames corresponds to replacing each rule $a \to w$ in a grammar with the rule $a \to w^{\mathsf{R}}$, where w^{R} is the word w written in reverse order, which, in turn, transforms a right-linear grammar into a left-linear one and vice versa. However, let us sketch an explicit filtration construction for left-linear grammars, since its ideas will be used later in Section 4.5.

Assume that Π is a left-linear grammar with converse terminals over $\Sigma \cup \overline{\Sigma}$, where $\Sigma = T \cup N$, so it consists of rules of the form $a \to cu$, where $a, c \in N$ and $u \in (T \cup \overline{T})^*$. To simplify notation, below we interpret $(\Sigma \cup \overline{\Sigma})$-formulas in Σ-frames and Σ-models in an obvious way.

Given a Π-model $M = (W, (R_e)_{e \in \Sigma}, V)$ and a finite Sub-closed set of formulas Γ over $\Sigma \cup \overline{\Sigma}$, we will build a Π-model that is a Γ-filtration of M. For this, we introduce the operators that are in a sense dual to the operators N, Π, S defined in Sections 4.2 and 4.3:

$$\begin{array}{rcl} \overline{N}(\Psi) & := & \mathsf{Sub}\left\{ [\overline{c}]\varphi \mid a, c \in N,\ [\overline{a}]\varphi \in \Psi \right\}, \\ \overline{\Pi}(\Psi) & := & \mathsf{Sub}\left\{ [\overline{v}]\varphi \mid [\overline{a}]\varphi \in \Psi,\ (a \to v) \in \Pi \right\}, \\ \overline{S}(\Psi) & := & \mathsf{Sub}\left\{ [\overline{a}]\neg[a]\varphi \mid [a]\varphi \in \Psi \right\}. \end{array}$$

Next we put $\Lambda = \Gamma \cup \overline{S}(\Gamma)$, $\Delta = \Lambda \cup \overline{N}(\Lambda)$, and $\Phi = \Delta \cup \overline{\Pi}(\Delta)$.

Now we set $\widehat{R}_e$ to be R_e^{min} if $e \in T$, and $\Delta_e^\sharp$ if $e \in N$. Then we prove an analogue of the Max-frame lemma: $\Delta_a^\sharp \supseteq \Delta_c^\sharp \circ \Phi_u^\sharp$, for each rule $(a \to cu) \in \Pi$. This allows us to prove that $\widehat{F} \models \Pi$, $\widehat{R}_e \subseteq \Gamma_e^\sharp$ and $\widehat{R}_e \subseteq \Gamma_e$, for all $e \in \Sigma$.

4.5 Bi-recursive grammar logics (with converse terminals)

We call a grammar *bi-recursive* if it consists of rules of two kinds: *right-recursive* $a \to ua$ and *left-recursive* $a \to av$, for $a \in N$ and $u, v \in T^*$. So these grammars combine right and left rules, but in every rule, the non-terminal in the body (ua or av) is always the same as in the head.

Let $\Pi = \Pi^r \cup \Pi^\ell$, where Π^r (resp., Π^ℓ) consists of rules of the form $a \to ua$ (resp., $a \to av$) with $a \in N$ and $u, v \in (T \cup \overline{T})^*$. The filtration for Π proceeds in two stages: first, we build the mini-maximal frame for Π^ℓ (as in Section 4.4) and then take its Π^r-closure (as in Section 4.2.2). In order for this construction to work, the sets of formulas Λ, Δ, Φ must be chosen properly.

Given a Π-model $M = (W, (R_e)_{e \in \Sigma}, V)$ and a finite Sub-closed set of $(\Sigma \cup \overline{\Sigma})$-formulas Γ, we will build a Π-model that is a Γ-filtration of M. Put

$$\Lambda = \Gamma \cup S(\Gamma), \qquad \Delta = \Lambda \cup \overline{S}(\Lambda), \qquad \Phi = \Delta \cup \Pi^r(\Delta) \cup \overline{\Pi^\ell}(\Delta).$$

Again, $\widehat{W} := W/{\sim_\Phi}$ and the valuation $\widehat{V}$ is canonical. It remains to build $\widehat{R}_e$.

Stage 1. Take $\mathbb{F} = (\widehat{W}, (\mathbb{R}_e)_{e \in \Sigma})$, where $\mathbb{R}_e$ is R_e^{min} if $e \in T$ and $\Delta_e^\sharp$ if $e \in N$. As in Section 4.4, we show that $(\mathbb{F}, \widehat{V})$ is a Π^ℓ-model and a Λ-filtration of M.

Stage 2. Take $\widehat{F} = (\widehat{W}, (\widehat{R}_e)_{e \in \Sigma})$ to be the Π^r-closure of $\mathbb{F}$.
As in Section 4.2.2, we show that $(\widehat{F}, \widehat{V})$ is a Γ-filtration of M. By Lemma 3.1(a), $\widehat{F}$ is a Π^r-frame. Finally, $\widehat{F}$ is a Π^ℓ-frame. In fact,[11] the Π^r-closure of a Π^ℓ-frame is again a Π^ℓ-frame. This follows from a simple fact: if a relation R satisfies $R \supseteq R \circ S$, then the relation $R' = Q \circ R$ satisfies $R' \supseteq R' \circ S$ as well.

Remark 4.9 To any grammar considered above, we can add rules of the form $a \to u$, where $a \in N$ and $u \in (T \cup \overline{T})^*$. The Π-closure construction also works for Π consisting of rules $a \to (ax)^k a$, where $a \in N$, $x \in T^*$, and $k \geq 0$.

[11] This does not generalize to arbitrary right-linear and left-linear grammars Π^r and Π^ℓ.

5 Logics that do not admit filtration

Here we give examples of logics that do not admit filtration. Perhaps, the logic **GL** is the simplest example: if a **GL**-model $M = (W, \prec, V)$ has an infinite descending chain $\ldots \prec x_2 \prec x_1 \prec x_0$, then any filtration of M has a reflexive point and hence is not a **GL**-model. Despite of this, **GL** has the FMP.

We will prove that the logics $\mathbf{K} + \Diamond\Box p \to \Box\Diamond p$ and $\mathbf{K}_2 + [a]p \to [b][a][b]p$ do not admit filtration by showing the undecidability of the global satisfiability problem. The former logic is known to be decidable and to have FMP, which is proved by embedding it into $\mathbf{K}\times\mathbf{K}$ and then appealing to the decidability and FMP for $\mathbf{K}\times\mathbf{K}$ shown in [10]. It is open whether the latter logic is decidable. Arguing similary, one can prove the same negative results for $[b][a]p \to [a][b]p$ and even for $\langle a\rangle[b]p \to [a]p$ ([16, p. 19]).

A *domino system* is a triple $\mathcal{D} = (D, H, V)$, where $D \neq \varnothing$ is a set of *tile types* and $H, V \subseteq D\times D$ are horizontal and vertical matching relations. We say that $\mathcal{D}$ *tiles* $\mathbb{N}\times\mathbb{N}$ if there exists a function $t\colon \mathbb{N}\times\mathbb{N} \to D$ such that, for all $i, j \in \mathbb{N}$, we have $(t(i,j), t(i+1,j)) \in H$ and $(t(i,j), t(i,j+1)) \in V$. The following *domino tiling* problem is known [3] to be undecidable: determine whether a given domino system $\mathcal{D}$ tiles $\mathbb{N}\times\mathbb{N}$; similarly for $\mathbb{N}\times\mathbb{Z}$ and $\mathbb{Z}\times\mathbb{Z}$.

Given a domino system $\mathcal{D}$, we define $\lambda^{\mathcal{D}} = \lambda_0 \wedge \lambda^H \wedge \lambda^V$, where

$$\begin{aligned}
\lambda_0 &= (\bigvee_{d\in D} q_d) \wedge \bigwedge_{d\neq d'} \neg(q_d \wedge q_{d'}),\\
\lambda^H &= \bigwedge_{d\in D} \big(q_d \to [h](\bigvee_{d'\in H(d)} q_{d'})\big),\\
\lambda^V &= \bigwedge_{d\in D} \big(q_d \to [v](\bigvee_{d'\in V(d)} q_{d'})\big).
\end{aligned}$$

Theorem 5.1 *The global satisfiability problem for the logic* $\mathbf{L} := \mathbf{K}_2 + [h]p \to [v][h][v]p$ *is undecidable, and hence* $\mathbf{L}$ *does not admit filtration.*

Consider a frame $G = (W, R_h, R_v)$, where $W = \mathbb{N}\times\mathbb{Z}$ and

$$\begin{aligned}
R_h &= \{((i,j),(i+1,j)) \mid \ i \in \mathbb{N},\ j \in \mathbb{Z}\},\\
R_v &= \{((i,j),(i,j-(-1)^i) \mid i \in \mathbb{N}, j \in \mathbb{Z}\}.
\end{aligned}$$

Let G' be the restriction of G to the sector $S = \{(i,j) \mid 0 \leq j < (2i+3)/4\}$, see Figure 1. Denote the formula $\xi = \langle h\rangle\top \wedge \langle v\rangle\top$.

Lemma 5.2 *For any* $\mathbf{L}$*-frame* F*, if* $F, \theta \models \xi$ *for some valuation* θ *on* F*, then there is a homomorphism* $f\colon G' \to F$*.*

Proof. 1) We set $f(0,0) = c$, for an arbitrary point c in F.

2) Suppose that f is already defined on $\{(i,j) \mid j < \frac{2i+3}{4}; i \leq i_0\}$ for some even $i_0 = 2k_0$. By ξ, there is a point a_0 in F with $F \models f(i_0,0)R_h a_0$. We set $f(i_0+1, 0) = a_0$. Now suppose that $f(i_0+1, j)$ is defined for some $j \leq k_0$. By ξ, there exists a point a_{j+1} in F with $F \models f(i_0,j)R_v a_{j+1}$. Set $f(i_0+1, j+1) = a_{j+1}$. If $j < k_0$, then by $F \models R_v \circ R_h \circ R_v \subseteq R_h$, we have $F \models f(i_0, j+1)R_h a_{j+1}$, and so f is still a homomorphism.

Now suppose that f is defined on (i_0+1, k_0+1). By ξ, there exists b_{k_0+1} in F such that $F \models f(i_0+1, k_0+1)R_h b_{k_0+1}$. We set $f(i_0+2, k_0+1) = b_{k_0+1}$.

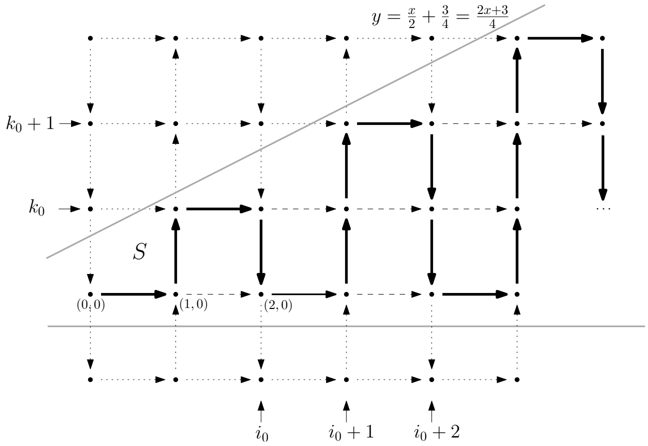

Fig. 1. The snake frame.

Suppose that $f(i_0 + 2, j + 1)$ is defined, for some $j \geq 0$. Then, by ξ, there is a point b_j in F such that $F \models f(i_0 + 2, j + 1)R_v b_j$. We set $f(i_0 + 2, j) = b_j$. Since $F \models R_v \circ R_h \circ R_v \subseteq R_h$, we have $F \models f(i_0 + 1, j)R_h b_j$ and so f is still a homomorphism. After $j = 0$ we have f defined on $\{(i, j) \mid j < \frac{2i+3}{4}; i \leq i_0 + 2\}$.

Iterating 2) yields the required homomorphism $f : G' \to F$. □

Proof. (of Theorem 5.1). Take a domino system $\mathcal{D}$. Without loss of generality, we can assume that

(tc) $\mathcal{D}$ tiles $\mathbb{N} \times \mathbb{Z}$ iff $\mathcal{D}$ tiles the sector S.

Indeed, recall that the tiling problem is undecidable because it models some Turing machine T. But after performing each instruction, the head of T can move to the right by at most one cell. This means that, in order to model T, it suffices to tile the sector $S' = \{(i, j) \mid i, j \in \mathbb{N},\ j \leq i\}$. If, additionally, without loss of generality, we assume that T idles after each right-move instruction, then we can model T by tiling the sector S. So, given a Turing machine T, we can generate a domino system $\mathcal{D}$ that satisfies (tc) — we only have to care how to tile $(\mathbb{N}\times\mathbb{Z}) \setminus S$ with special blank tiles without affecting the work of T.

Assuming (tc), we can show that $\xi \wedge \lambda^{\mathcal{D}}$ is globally **L**-satisfiable iff $\mathcal{D}$ tiles $\mathbb{N} \times \mathbb{Z}$. ($\Rightarrow$) If $\mathcal{D}$ tiles $\mathbb{N} \times \mathbb{Z}$, then we can use this tiling to define a model for $\xi \wedge \lambda^{D}$ based on G. ($\Leftarrow$) Suppose that $M \models \xi \wedge \lambda^{\mathcal{D}}$. Using Lemma 5.2, we can construct a homomorphism $f : G' \to M$. Now, from M we can read off a $\mathcal{D}$-tiling of S, and by (tc) conclude that there is a $\mathcal{D}$-tiling of $\mathbb{N}\times\mathbb{Z}$. □

Corollary 5.3 *The grammar logic corresponding to the context-free grammar* $\{h \to vhv,\ u \to uh,\ u \to uv\}$ *is undecidable.*

Proof. For every modal formula φ containing only $[h]$ and $[v]$, we have: φ is globally satisfiable in the logic $\mathbf{K}\{h \to vhv\}$ iff the formula $\langle u\rangle\top \wedge [u]\varphi$ is (locally) satisfiable in the logic $\mathbf{K}\{h \to vhv,\ u \to uh,\ u \to uv\}$. □

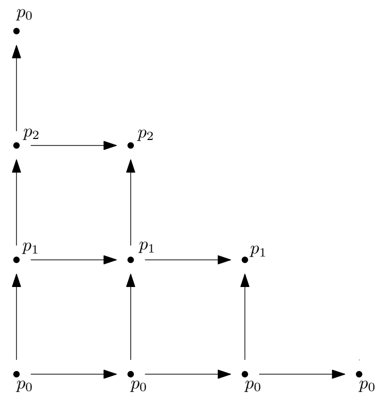

Fig. 2. A fragment of a "grid" model.

This gives a simple example of an undecidable context-free grammar logic without converse. Compare this with the undecidability of **PDL** enriched with the language $\{a^n b a^n \mid n \in \mathbb{N}\}$ shown in [12]. However, their formulas essentially use the **PDL**-constructs, and it is not clear if the latter can be eliminated so that the undecidability proof worked for context-free grammar logics.

Theorem 5.4 *The global satisfiability for* $\mathbf{K} + \Diamond\Box p \to \Box\Diamond p$ *is undecidable.*

Proof. Let $M = (W^G, R^G, \theta^G)$ be an infinite Kripke model based on a frame G (see Figure 2), where $W^G = \mathbb{N} \times \mathbb{N}$, $\theta^G(p_i) = \{(m,n) \mid n \equiv i \,(\mathrm{mod}\, 3)\}$, and

$$R^G = \{((m,n),(m+1,n)) \mid m,n \in \mathbb{N}\} \cup \{((m,n),(m,n+1)) \mid m,n \in \mathbb{N}\}.$$

Let ξ be the conjunction of the following formulas (which capture some properties of G) for $0 \le i \ne j \le 2$ (subscripts of p's are understood modulo 3):

$$\begin{array}{ll} (X1) & p_i \wedge p_j \to \bot \\ (X2) & p_i \to \Box(p_i \vee p_{i+1}) \\ (X3) & p_i \to (\Diamond p_i \wedge \Diamond p_{i+1}) \\ (X4) & p_0 \vee p_1 \vee p_2 \end{array}$$

Lemma 5.5 *Assume that* $F \models \Diamond\Box p \to \Box\Diamond p$. *If* $F, \theta^F \models \xi$ *for some* θ^F, *then there is a homomorphism* $f\colon G \to F$ *in the following sense:*

$$\begin{array}{ll} (\mathsf{Homo}) & \textit{if } G \models xRy \textit{ then } F \models f(x)Rf(y) \textit{ and} \\ & x \in \theta^G(p_i) \ \textit{ iff } \ f(x) \in \theta^F(p_i). \end{array}$$

Proof. Denote $N = (F, \theta^F)$. *Step 0.* From $(X3)$ and $(X4)$ it follows that there exists a point x in F such that $N, x \models p_0$. We set $f(0,0) = x$.

Step n. Suppose that, for some n, f is defined on $\{(i,j) \mid i,j \ge 0;\ i+j \le n\}$ and satisfies (Homo). We extend f to $\{(i,j) \mid i,j \ge 0;\ i+j = n+1\}$ as follows. By (X3), there is a point x in F such that $F \models f(n,0)Rx$ and $N, x \models p_0$. We set $f(n+1,0) = x$. Similarly, there is y in F such that $F \models f(0,n)Ry$ and $N, y \models p_{n+1}$. We set $f(0,n+1) = y$. Since $F \models \Diamond\Box p \to \Box\Diamond p$, the fact that, by I.H., for all i,j such that $i+j = n-1$, $F \models f(i,j)Rf(i,j+1)$

and $F \models f(i,j)Rf(i+1,j)$, implies that there exist points z_{ij} in F such that $F \models f(i,j+1)Rz_{ij} \wedge f(i+1,j)Rz_{ij}$. Now we define $f(i+1,j+1) = z_{ij}$. Then (X2) and I.H. imply that $N, z_{ij} \models p_j$. We claim that f is now defined on $\{(i,j) \mid 1 \leq i+j \leq n+1\}$ and satisfies (Homo). □

For a domino system $\mathcal{D} = (D, H, V)$, we define $\lambda^{\mathcal{D}} = \lambda_0 \wedge \lambda^H \wedge \lambda^V$, where

$$\lambda_0 = (\bigvee_{d \in D} q_d) \wedge \bigwedge_{d \neq d'} \neg(q_d \wedge q_{d'}),$$
$$\lambda^H = \bigwedge_{0 \leq i \leq 2} \bigwedge_{d \in D} \big(p_i \wedge q_d \to \Box(p_i \to \bigvee_{d' \in H(d)} q_{d'})\big),$$
$$\lambda^V = \bigwedge_{0 \leq i \leq 2} \bigwedge_{d \in D} \big(p_i \wedge q_d \to \Box(p_{i+1} \to \bigvee_{d' \in V(d)} q_{d'})\big).$$

We claim that $\xi \wedge \lambda^{\mathcal{D}}$ is globally satisfiable on some frame F validating $\Diamond\Box p \to \Box\Diamond p$ iff $\mathcal{D}$ tiles $\mathbb{N} \times \mathbb{N}$. This gives us the desired reduction. □

6 Conclusion and further research

In this paper we investigate classes of frames and modal logics that admit filtration and operations on them that preserve this property (they are called filtration safe). On the one hand, this notion is useful for obtaining results on the FMP and the decidability of logics. On the other, it appears robust, for many interesting operations turn out to be filtration safe (see Section 2).

We used filtration safe operations to show that every regular grammar logic (with converse) admits filtration and hence is decidable (Theorem 3.3). Note that all known examples of undecidable grammar logics (see e.g. Corollary 5.3) correspond to irregular grammars. The following question arises naturally.

Open problem. *Are the following claims equivalent, for every grammar* Π*:*

(i) Π *is a regular grammar,*
(ii) *the logic* $\mathbf{K}\Pi$ *is decidable,*
(iii) *the logic* $\mathbf{K}\Pi$ *admits filtration,*
(iv) *the logic* $\mathbf{K}\Pi$ *has the finite model property,*
(v) *the logic* $\mathbf{K}\Pi$ *has the exponential model property?*

The approach taken in this paper could be developed in various directions. One can examine other interesting operations on relations for filtration safety. Note that all operations on classes of frames considered above are in fact induced by operations on frames. It is reasonable to consider more general operations. For example, the following operation is filtration safe: given a class $\mathcal{F}$ of frames of the form (W, R), build the class $\{(W, R, S) \mid (W, R) \in \mathcal{F},\, R \subseteq S\}$. One could also seek for general conditions on operations that are sufficient for filtration safety; Lemma 1.6 gives an idea what such conditions might look like.

A 'relativized' notion also makes sense: one can say that a logic $\mathbf{L}$ *admits filtration in a class of frames* $\mathcal{F}$ if $\mathbf{L}$ is the logic of $\mathcal{F}$ and $\mathcal{F}$ admits filtration. As we mentioned in Example 1.4, the logics **S4.2** and **S4.3** admit filtration in the classes of their point-generated frames. Of course, in order to use the relativized notion for obtaining decidability results, we need to assume that the subclass of finite frames of the class $\mathcal{F}$ is recognizable.

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