Chronological future modality in Minkowski spacetime

Ilya Shapirovsky and Valentin Shehtman

1 Introduction

The problem of logical foundations of contemporary physics was included by David Hilbert in the list of the most important mathematical problems and generated an interesting research area in nonclassical logic. Study of relativistic temporal logics is a natural topic within this area. Their investigation was initiated by Arthur Prior's proposal in [10], but at the early stage did not move fast — perhaps because relativistic time is both branching and dense, which is rather unusual for modal logic.

Let us recall that two basic relations in Minkowski spacetime are causal ($\prec$) and chronological ($\preceq$) accessibility. The causal future of a point-event x consists of all those points y, to which a signal from x can be sent; $x \prec y$ if this signal is slower than light.

The first significant result in relativistic temporal logic was the theorem by Robert Goldblatt [6] identifying the ("Diodorean") modal logic of relation $\preceq$ as the well-known **S4.2**. Then in [12] the second author described modal logics of domains on Minkowski plane ordered by $\preceq$.

This paper makes the next essential step after the past twenty years. It solves one of three problems put by R. Goldblatt in [6] (see also [7]): to axiomatize the modal logic of the frame $(\mathbb{R}^n, \prec)$.

For this logic $\mathbf{L}_2$ we present an axiom system. Its axioms are widely known in modal logic, except for the specific axiom of 2-density, first introduced in [6]. The logic $\mathbf{L}_2$ is rather standard, but the completeness proof for the intended interpretation is not so straightforward. The main technical problem is the proof of the finite model property. As 2-density is not preserved under filtration in the Lemmon–Segerberg style (when some worlds are identified), we use the Kripke–Gabbay method of selective filtration instead. This method allows us to extract a finite submodel from an infinite model, see e.g. [2]. In our case selective filtration is applied to the canonical model in a combination with the method of maximal points, due to Kit Fine [3]. Similar arguments were used for various modal and intermediate logics by P. Miglioli, M. Zakharyaschev, F. Wolter and others. The finite model

Advances in Modal Logic, Volume 4, 437–459.

characterization is convenient for obtaining complexity bounds of $\mathbf{L}_1$, $\mathbf{L}_2$; this subject is postponed until a further publication.

The final part of the proof is a geometric construction of a p-morphism following the lines of [6] and [12].

In the last Section we discuss applications of our results to many-dimensional modal logics, such as products and interval logics.

2 Basic notions

In this paper all *modal logics* are normal monomodal propositional logics containing $\mathbf{K4}$; as usual, modal logics are considered as certain sets of formulas. For a modal logic Λ and a modal formula A, the notation $\Lambda \vdash A$ means $A \in \Lambda$; $\Lambda + A$ denotes the smallest modal logic containing $\Lambda \cup \{A\}$.

We assume that $\Diamond$, $\to$, $\bot$ are the basic connectives, and $\Box$, $\neg$, $\vee$, $\wedge$, $\top$ are derived. PV denotes the set of propositional variables.

Here are the names for some particular modal formulas:

$$A4 := \Diamond\Diamond p \to \Diamond p, \qquad AD := \Diamond\top,$$
$$A2 := \Diamond\Box p \to \Box\Diamond p, \qquad Ad := Ad_1 = \Diamond p \to \Diamond\Diamond p,$$
$$Ad_n := \Diamond p_1 \wedge \ldots \wedge \Diamond p_n \to \Diamond(\Diamond p_1 \wedge \ldots \wedge \Diamond p_n),$$

and the names for some modal logics:

$$\mathbf{K4} := \mathbf{K} + A4, \qquad \mathbf{D4} := \mathbf{K4} + AD,$$
$$\mathbf{D4.2} := \mathbf{D4} + A2, \qquad \mathbf{L}_1 := \mathbf{D4d}_2 := \mathbf{D4} + Ad_2,$$
$$\mathbf{L}_2 := \mathbf{D4.2d}_2 := \mathbf{L}_1 + A2.$$

By *(Kripke) frame* we mean a non-empty set with a transitive relation (W, R). A *(Kripke) model* is a Kripke frame with a valuation: $M = (W, R, \theta)$, where $\theta :\ PV \longrightarrow 2^W$ (2^W denotes the power set of W).

For a Kripke model $M = (W, R, \theta)$, the notation $x \in M$ means $x \in W$. The sign $\models$ denotes the truth at a point of a Kripke model and also the validity in a Kripke frame.

For a class of frames $\mathcal{F}$, $\mathbf{L}(\mathcal{F})$ denotes the *modal logic determined by* $\mathcal{F}$, i.e., the set of all formulas that are valid in all frames from $\mathcal{F}$. For a single frame F, $\mathbf{L}(F)$ abbreviates $\mathbf{L}(\{F\})$. Recall that a *cluster* in (W, R) is an equivalence class under the relation

$$\sim_R := (R \cap R^{-1}) \cup I_W$$

(where I_W is the equality relation on W), a *degenerate cluster* is an irreflexive singleton. As usual, for $x \in W$, $V \subseteq W$ we denote $R(x) := \{y \mid xRy\}$, $R(V) := \bigcup\limits_{x \in V} R(x)$. A cluster is called *maximal* if $R(C) \subseteq C$; a point

$x \in W$ is called *maximal* if its cluster is maximal. The associated relation between clusters

$$C \leq_R D := D \subseteq R(C),\ C <_R D := C \leq_R D \text{ and } C \neq D$$

are transitive and antisymmetric, and $<_R$ is irreflexive. For a frame $F = (W, R)$ let $F/\sim_R := (W/\sim_R, \leq_R)$.

A Kripke model $M_1 = (W_1, R_1, \theta_1)$ is a *(weak) submodel* of $M = (W, R, \theta)$ (notation: $M_1 \subseteq M$) if $W_1 \subseteq W$, $R_1 \subseteq R$, $\theta_1(q) = \theta(q) \cap 2^{W_1}$ for every $q \in PV$. A particular case of a submodel is when $R_1 = R \cap (W_1 \times W_1)$; in this case M_1 is called a *restriction of* M *to* W_1 and denoted by $M|W_1$. A submodel $M|W_1$ is called *generated* if $R(W_1) \subseteq W_1$. A particular case of a generated submodel is a *cone* $M^x = M|W^x$, where $W^x = \{x\} \cup R(x)$.

It is well-known that formula $A4$ corresponds to transitivity of a Kripke frame and AD corresponds to seriality: $\forall x \exists y\ xRy$. The correspondents for $A2$, Ad_n are also easily described:

LEMMA 1. *For a frame F,*

- $F \vDash A2$ *iff* F *is* confluent, *i.e., satisfies*

$$\forall x \forall y_1 \forall y_2 \exists z (xRy_1 \ \&\ xRy_2 \Rightarrow y_1Rz \ \&\ y_2Rz);$$

- $F \vDash Ad_n$ *iff* F *is* n-dense, *i.e., the following holds:*

$$\forall x \forall y_1 \ldots \forall y_n \exists z (xRy_1 \ \&\ \ldots \ \&\ xRy_n \Rightarrow xRz \ \&\ zRy_1 \ \&\ \ldots \ \&\ zRy_n).$$

By Sahlqvist Theorem (cf. [2]), we also have completeness:

PROPOSITION 2. *The logics* $\mathbf{D4.2}$, $\mathbf{L_1}$, $\mathbf{L_2}$ *are canonical.*

So we obtain

PROPOSITION 3. $\mathbf{L}_1$ *is determined by the class of all serial 2-dense frames,* $\mathbf{L}_2$ *is determined by the class of all serial confluent 2-dense frames.*

LEMMA 4. $\mathbf{K4} + Ad_2 \vdash Ad_n$ *for all* n.

Proof. It is clear that $\mathbf{K4} + Ad_2 \vdash Ad_1$. A syntactic proof of $\mathbf{K4} + Ad_2 \vdash Ad_n$ for $n > 2$ is rather easy and is left for the reader. Another proof is based on completeness theorem for $\mathbf{K4} + Ad_n$ and the observation that for a transitive relation, 2-density implies n-density. ∎

3 The finite model property

Let us first recall a simple lemma on selective filtrations.

DEFINITION 5. Let M be a Kripke model, Ψ a set of formulas closed under subformulas.

A submodel $M_1 \subseteq M$ (with the relation R_1) is called a *selective filtration of M through Ψ* if for any $x \in M_1$, for any formula A

$$\Diamond A \in \Psi \ \&\ M, x \vDash \Diamond A \Rightarrow \exists y \in R_1(x)\ M, y \vDash A.$$

LEMMA 6. *If M_1 is a selective filtration of M through Ψ, then for any $x \in M_1$, for any $A \in \Psi$*

$$M, x \vDash A \Leftrightarrow M_1, x \vDash A.$$

Proof. By induction on the length of A. We consider only the case $A = \Diamond B$.

If $M, x \vDash \Diamond B$, then by Definition 5, $M, y \vDash B$ for some $y \in R_1(x)$. So $M_1, y \vDash B$ by the induction hypothesis, and thus $M_1, x \vDash \Diamond B$.

Conversely, if $M_1, x \vDash \Diamond B$, then $M_1, y \vDash B$ for some $y \in R_1(x) \subseteq R(x)$. Hence $M, y \vDash B$ by the induction hypothesis, and thus $M, x \vDash \Diamond B$. ■

Now let us prove some useful properties of canonical models. First we recall the maximality property of canonical models, cf. [3], [2]:

LEMMA 7. *Let $\mathfrak{M}$ be the canonical model of a modal logic Λ, and assume that a formula B is satisfied in some $x \in \mathfrak{M}$. Consider the set of all those clusters in $\mathfrak{M}^x$, in which B is satisfied:*

$$\Gamma := \{C \subseteq \mathfrak{M}^x \mid \exists y \in C\ B \in y\}.$$

Then $\mathfrak{M}|\bigcup\Gamma$ contains a maximal cluster (with respect to the relation $\leq_R$).

Proof. By assumption, $\Gamma \neq \varnothing$, and so due to Zorn Lemma, it suffices to check that every $\leq_R$-chain Σ of clusters from Γ has an upper bound in Γ. Let

$$S := \{A \mid \exists C \in \Sigma\ \forall y \in C\ \Box A \in y\} \cup \{B\},$$

and consider two cases.

(1) Assume that S is $\mathbf{\Lambda}$-inconsistent. Then there exist clusters $C_1, \ldots, C_n \in \Sigma$ and points $y_1 \in C_1, \ldots, y_n \in C_n$ such that $B \in y_i$ for every i, and for some formulas $\Box A_1 \in y_1, \ldots, \Box A_n \in y_n$ we have

$$(*) \quad \neg(A_1 \wedge \ldots \wedge A_n \wedge B) \in \mathbf{\Lambda}.$$

Since Σ is a chain, $C_1, \ldots, C_n$ are $\leq_R$-comparable, and we may assume that C_1 is the $\leq_R$-greatest among them. So for every i, $y_i R y_1$ or $y_i = y_1$. But in the canonical model we have $y_i \vDash \Box A_i$, and thus we obtain $y_1 \vDash \Box A_1 \wedge \ldots \wedge \Box A_n$. Then it follows that C_1 is an upper bound of Σ. In fact, otherwise for some $y_0 \in R(y_1)$ we have $y_0 \vDash A_1 \wedge \ldots \wedge A_n \wedge B$, which contradicts (*).

(2) Now assume that S is $\mathbf{\Lambda}$-consistent. Then by Lindenbaum Lemma, $S \subseteq z$ for some $z \in \mathfrak{M}$. Let Z be the cluster of z. By definition of S, $Z \in \Gamma$, and for any $y \in \bigcup \Sigma$ we have yRz. Thus Z is an upper bound of Σ. ∎

LEMMA 8. *Let Λ be a modal logic containing $\mathbf{L}_1$, $\mathfrak{M}$ a canonical model of $\mathbf{\Lambda}$. Let $x \in \mathfrak{M}$, and assume that $\Diamond A_1, \ldots, \Diamond A_n \in x$. Let*

$$Y := \{y \mid xRy \ \&\ \mathfrak{M}, y \vDash \Diamond A_1 \wedge \ldots \wedge \Diamond A_n\}.$$

Then $\mathfrak{M}|Y$ contains a maximal cluster, which is non-degenerate.

Proof. Let $B := \Diamond A_1 \wedge \ldots \wedge \Diamond A_n$. By Lemma 4, $Ad_n \in \mathbf{L}_1$, and thus $\mathfrak{M} \vDash B \to \Diamond B$. So we have $\mathfrak{M}, x \vDash \Diamond B$, which implies $Y \neq \varnothing$ (Fig. 1).

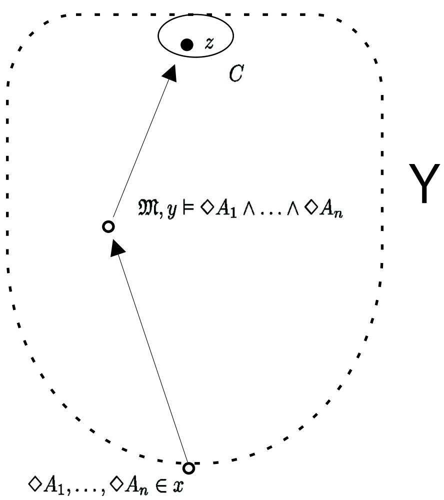

Figure 1.

By Lemma 7, $\mathfrak{M}|Y$ has a maximal cluster C. Since $\mathfrak{M} \vDash B \to \Diamond B$, for any $z \in C$ we have $\mathfrak{M}, z \vDash \Diamond B$, and so there exists $t \in Y$ such that zRt. But $t \in C$, by maximality of C. Therefore, C is non-degenerate. ∎

One can easily check the following

LEMMA 9. *A finite frame $F = (W, R)$ is 2-dense iff for any irreflexive $x \in W$ with $R(x) \neq \varnothing$, there exists a unique successor cluster, which is non-degenerate.*

Let $\mathcal{F}_1$ be the class of all finite 2-dense serial frames, and let

$$\mathcal{F}_2 := \{F + C \mid F \in \mathcal{F}_1, C \text{ is a finite non-degenerate cluster}\},$$

where + denotes the ordinal sum. Then we have

THEOREM 10. $\mathbf{L}_1 = \mathbf{L}(\mathcal{F}_1), \quad \mathbf{L}_2 = \mathbf{L}(\mathcal{F}_2)$.

Proof. (I) First consider $\mathbf{L}_1$. Let μ_1 be the class of Kripke models over frames from $\mathcal{F}_1$. Given an $\mathbf{L}_1$-consistent formula B, we have to construct a a model $M \in \mu_1$, where B is satisfied.

Let $\mathfrak{M}$ be the canonical model of $\mathbf{L}_1$, R the accessibility relation in $\mathfrak{M}$. Let us say that $y \in \mathfrak{M}$ *realizes* a formula $\Diamond A$ if $\mathfrak{M}, y \vDash A$ (i.e., $A \in y$). Since B is consistent, we have $\mathfrak{M}, x_0 \vDash B$ for some x_0. Let Ψ be the set of all subformulas of B,

$$S := \{\Diamond A \mid \Diamond A \in \Psi\} \cup \{\Diamond \top\}.$$

For $x, y \in \mathfrak{M}$ let

$$x \approx y := (S \cap x = S \cap y).$$

Note that xRy implies $S \cap y \subseteq S \cap x$, and thus $x \sim_R y$ implies $x \approx y$.

For $x \in \mathfrak{M}$ let

$$Y_x := \{y \in R(x) \mid x \approx y\}.$$

Then Y_x is a union of clusters.

A point $x \in \mathfrak{M}$ is called *critical* if $\forall y \in R(x)$ $(x \approx y \Rightarrow x \sim_R y)$, i.e., if the cluster of x is maximal in Y_x. By Lemma 8, for any $x \in \mathfrak{M}$ there exists a maximal and non-degenerate cluster in Y_x; so we can choose a critical point x' in Y_x. We also assume that $x' = x$ if x already is critical; thus $x'' = x'$ for every x.

Now we construct a required M by induction.

Stage 0. Put $M_0 := \mathfrak{M}|\{x_0, x_0'\}$.

If $S = \{\Diamond \top\}$, the construction terminates at this stage, i.e., we take $M = M_0$. Trivially, M_0 is a selective filtration of $\mathfrak{M}$ through Ψ, and so $M_0, x_0 \vDash B$ by Lemma 6. $M_0 \in \mu_1$, since x_0' is reflexive (in this case its cluster is maximal in $\mathfrak{M}$).

If $S \neq \{\Diamond \top\}$, we store the new point in the set: $X_0 := \{x_0'\}$, and continue the construction.

Stage (n+1). Assume that at stage n we have a model $M_n \subseteq \mathfrak{M}$, $M_n \in \mu_1$, and a non-empty subset $X_n \subseteq M_n$ such that for every $x \in M_n$ the following holds.

(1) $x' \in M_n$;

(2) if $x \in X_n$, then x is critical;

(3) if $x' \notin X_n$, $\Diamond A \in S \cap x$, then $\exists y \in R_n(x')\ A \in y$,

where R_n is the accessibility relation in M_n. Every formula $\Diamond A \in S \cap x$ is called *essential* (for x). For a critical x, let $S_1(x)$ be the set of all essential formulas that are realizable in the cluster of x:

$$S_1(x) := \{\Diamond A \in S \cap x \mid \exists t \sim_R x\ A \in t\},$$

and let

$$S_2(x) := (S \cap x) - S_1(x).$$

Obviously, $S_1(x) \neq \varnothing$, because $\top \in x$. Now for every $x \in X_n$ we proceed as follows.

If $S_1(x) = \{\Diamond A_1, \ldots, \Diamond A_m\}$, then for $i = 1, \ldots, m$ we choose $t_i \sim_R x$ such that $A_i \in t_i$. Of course, it may happen that $t_i \in M_n$ or $t_i = t_j$ for some different i, j.

Similarly, if $S_2(x) = \{\Diamond B_1, \ldots, \Diamond B_k\}$, then for $i = 1, \ldots, k$ we choose $z_1, \ldots, z_k \in R(x)$ such that $B_i \in z_i$. Let
$U_x := \{z_1, \ldots, z_k\}$ or $U_x = \varnothing$ (if $S_2(x) = \varnothing$),
$U'_x := \{z' \mid z \in U_x\}$,
$W_x := U_x \cup U'_x \cup \{t_1, \ldots, t_m\}$ (Fig. 2).

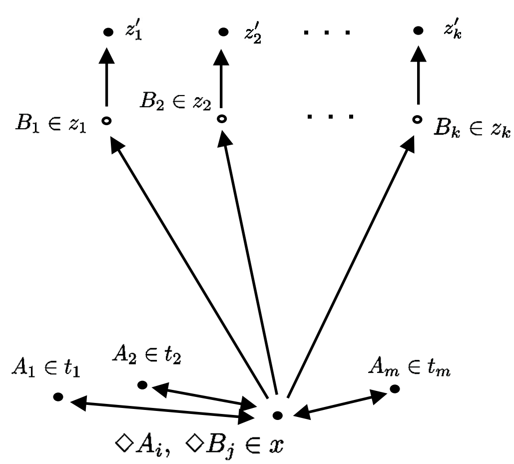

Figure 2.

We define M_{n+1} as the submodel of $\mathfrak{M}$ obtained by adding all points of $\bigcup\limits_{x\in X_n} W_x$ to M_n. The relation in M_{n+1} is defined as follows.

- If a is reflexive then $R_{n+1}(a) := R(a) \cap M_{n+1}$.
- If a is irreflexive then $R_{n+1}(a) := R(a') \cap M_{n+1}$.

One can easily check that R_{n+1} is transitive; note that for irreflexive a, $aR_{n+1}bR_{n+1}c$ implies $a'RbRc$, and thus $a'Rc$, i.e., $aR_{n+1}c$.

R_{n+1} is serial, since $aR_{n+1}a'$.

By Lemma 9, R_{n+1} is 2-dense; in fact, if a is irreflexive, then the cluster of a' is the first in $R_{n+1}(a)$. Also note that $R_n \subseteq R_{n+1}$, since $R_n(a)$ is defined in the same manner as $R_{n+1}(a)$, and the case $n = 0$ is not exceptional.

Thus $M_{n+1} \in \mu_1$.

Let X_{n+1} be the set of new critical points:

$$X_{n+1} := \bigcup_{x\in X_n} U'_x.$$

Then properties (1)-(3) hold in M_{n+1}. In fact, (1), (2) are obvious.

To check (3), suppose $x \in M_{n+1}$, $x' \notin X_{n+1}$. Then by construction, $x \in M_n$. Since (3) holds for M_n and $R_n \subseteq R_{n+1}$, it remains to consider the case $x' \in X_n$. But then by construction, every essential formula is realizable in M_{n+1} (in one of the points $t_1, \dots, t_m, z_1 \dots, z_k$).

If $X_{n+1} = \varnothing$, the construction terminates at this stage, and we put $M := M_{n+1}$.

According to the construction, if $x \in X_n$, $\Diamond B_i \in S_2(x)$, we have $z_i \not\sim_R x$, and thus $S \cap z'_i = S \cap z_i \subset S \cap x$, because x is critical. So the number of essential formulas decreases at every step, more precisely:

$$\max\{|S \cap x| \mid x \in X_{n+1}\} < \max\{|S \cap x| \mid x \in X_n\}.$$

Therefore the number of steps in the construction does not exceed the cardinality of S, and we obtain a finite model M.

Due to property (3), the resulting model M is a selective filtration of $\mathfrak{M}$ through Ψ. Since $\mathfrak{M}, x_0 \vDash B$, we obtain $M, x_0 \vDash B$ by Lemma 6.

(II) Similarly, in the case of $\mathbf{L}_2$, consider the canonical model $\mathfrak{M}$, an $\mathbf{L}_2$-consistent formula B; take a world $x \in \mathfrak{M}$, where B is satisfied. By Lemma 8, the cone $\mathfrak{M}^x$ has a maximal and non-degenerate cluster C. By Lemma 1 and Proposition 2, $\mathfrak{M}$ is confluent, and thus C is its final cluster.

As we are interested only in subformulas of B, consider the equivalence relation on C:

$$x \approx y := (\Psi \cap x = \Psi \cap y)$$

and take a finite cluster $C_1 \subseteq C$ containing all representatives under $\approx$. Then the submodel $M' \subseteq \mathfrak{M}$ obtained by adding the cluster C_1 to M (constructed in the proof of (I)) is in μ_1. In fact, the successor cluster of an irreflexive x in M also fits for M'.

To show that M' is a selective filtration of $\mathfrak{M}$ through Ψ, first note that for $x \in M$, all essential formulas can be realized within M. And for $x \in C_1$, if $\Diamond A \in x \cap \Psi$, then $A \in y$ for some $y \in C$ (since C is maximal), and by the choice of C_1, there exists $y_1 \approx y$ in C_1.

Thus B is satisfiable in M'.

Let F' be the frame of M', and put the copy of C_1 on the top of F'. The new frame F'' is in $\mathcal{F}_2$, since $F' \in \mathcal{F}_1$. We also obtain that B is satisfiable in F'', since it is satisfiable in F', and there exists an obvious p-morphism from F'' onto F', which identifies two copies of C_1.

Therefore we obtain the inclusion $\mathbf{L}(\mathcal{F}_2) \subseteq \mathbf{L}_2$. The converse follows easily, because every frame in $\mathcal{F}_2$ is confluent, serial and 2-dense. ■

DEFINITION 11. A *reflexive tree* is a rooted poset (W, R), in which every subset $R^{-1}(x)$ is a chain; a *tree* (in this paper) is frame, whose reflexive closure is a reflexive tree. A frame F is called a *quasitree* if its cluster frame $F/\sim_R$ is a tree.

Now let $\mathcal{G}_1$ be the class of all finite 2-dense serial quasitrees, and let

$$\mathcal{G}_2 := \{F + C \mid F \in \mathcal{G}_1,\ C \text{ is a finite non-degenerate cluster}\}.$$

We can give more specific characterizations of $\mathbf{L}_1$, $\mathbf{L}_2$:

LEMMA 12. $\mathbf{L}_1 = \mathbf{L}(\mathcal{G}_1)$*;* $\mathbf{L}_2 = \mathbf{L}(\mathcal{G}_2)$.

Proof. The inclusions $\mathcal{G}_1 \subseteq \mathcal{F}_1$ and $\mathcal{G}_2 \subseteq \mathcal{F}_2$ imply $\mathbf{L}(\mathcal{G}_1) \supseteq \mathbf{L}_1$, $\mathbf{L}(\mathcal{G}_2) \supseteq \mathbf{L}_2$. Thus to prove the first statement of Lemma, it is sufficient to show that every rooted frame $H = (W, R) \in \mathcal{F}_1$ is a p-morphic image of some frame from $\mathcal{G}_1$. First let us consider the cluster frame $H_1 = H/\sim_R$. By standard unravelling argument, we can present H_1 as a p-morphic image of a finite tree H' consisting of all paths in H_1, cf. [2]. In more detail, $H' = (W', R')$, where

$W' := \{(C_0, \ldots, C_k) \mid C_0 <_R C_1 <_R \cdots <_R C_k\}$,

C_0 is the initial cluster in H;

$\alpha R' \beta$ iff either β is a continuation of α and $\alpha \neq \beta$, or $\alpha = \beta$ and the last cluster of α is non-degenerate.

Then the map $f : (C_0, \ldots, C_k) \mapsto C_k$ is a p-morphism $H' \twoheadrightarrow H_1$. Now consider a quasitree $H^* = (W^*, R^*)$, in which

$$W^* := \{(C_0, \ldots, C_k, w) \mid (C_0, \ldots, C_k) \in W',\ w \in C_k\},$$
$$(C_0, \ldots, C_k, w) R^* (C_0, \ldots, C'_l, v) \text{ iff } (C_0, \ldots, C_k) R' (C_0, \ldots, C'_l).$$

Note that $H^* \in \mathcal{G}_1$, and we obtain a p-morphism $g : H^* \twoheadrightarrow H$ such that $g(C_0, \dots, C_k, w) = w$.

Next, for any finite non-degenerate cluster C, we can extend g to a p-morphism $g' : H^* + C \twoheadrightarrow H + C$. Since $H^* + C \in \mathcal{G}_2$, this yields the second statement. ■

4 Further completeness results

Let $T_{\mathbb{Z}}$ be the set of finite sequences of integers, T_n the set of finite sequences of numbers $\{1, 2, \dots, n\}$. The sequences are ordered in a standard way: $\sigma_1 \sqsubseteq \sigma_2$ iff σ_1 is an initial part of σ_2. Let $\sigma_1 \sqsubset \sigma_2$ iff $\sigma_1 \sqsubseteq \sigma_2$ and $\sigma_1 \neq \sigma_2$; $\mathbf{T}_{\mathbb{Z}} := (T_{\mathbb{Z}}, \sqsubseteq)$, $\mathbf{T_2} := (T_2, \sqsubseteq)$.

λ denotes the empty sequence; $\sigma_1\sigma_2$ denotes the concatenation of sequences σ_1 and σ_2.

Now let us extend the tree $\mathbf{T_2}$ by inserting an extra irreflexive point at every edge. More precisely, this means the following.

For a non-empty $\tau \in T_2$ let $\tau^- := (\tau, 0)$. Consider the set

$$I_2 := T_2 \cup \{\tau^- \mid \tau \in T_2,\ \tau \neq \lambda\}.$$

Let $\lessdot$ be the minimal transitive relation on I_2 satisfying the conditions:

- $\lessdot | T_2 = \sqsubseteq$;
- if $\sigma \in T_2$, $i \in \{0, 1\}$, $\tau = \sigma i$, then $\sigma \lessdot \tau^- \lessdot \tau$.

Let $\mathbf{I_2} := (I_2, \lessdot)$, $\mathbf{I_2^+} := \mathbf{I_2} + C_{\mathbb{Z}}$, where $C_{\mathbb{Z}}$ is a countable cluster (Fig. 3). Since $\mathbf{I_2}$ is 2-dense, we have

$\mathbf{L}(\mathbf{I_2}) \supseteq \mathbf{L_1},\ \mathbf{L}(\mathbf{I_2^+}) \supseteq \mathbf{L_2}$.

THEOREM 13. $\mathbf{L}(\mathbf{I_2}) = \mathbf{L}_1,\ \mathbf{L}(\mathbf{I_2^+}) = \mathbf{L}_2$.

Proof. For a frame F let $F_0 := E + F$, where E is a reflexive singleton. For $i = 1, 2$ let $\mathcal{K}_i := \{F_0 \mid F \in \mathcal{G}_i\}$. Then $\mathbf{L}_i = \mathbf{L}(\mathcal{K}_i)$. In fact, $\mathcal{K}_i \subseteq \mathcal{G}_i$ implies $\mathbf{L}_i \subseteq \mathbf{L}(\mathcal{K}_i)$ (by Lemma 12). On the other hand, for any $F \in \mathcal{G}_i$ there exists a p-morphism $F_0 \twoheadrightarrow F$ sticking E to the initial cluster of F (which is non-degenerate); therefore $\mathbf{L}_i \subseteq \mathbf{L}(\mathcal{K}_i)$.

Now to prove the inclusion $\mathbf{L}(\mathbf{I_2}) \subseteq \mathbf{L}_1 = \mathbf{L}(\mathcal{K}_1)$, let us construct a p-morphism from $\mathbf{I_2}$ onto an arbitrary frame $G = (W, R) \in \mathcal{K}_1$.

Let F be the restriction of G to the set of reflexive points. It is well-known [6], [12] that there exists a p-morphism $f : \mathbf{T_2} \twoheadrightarrow F$.

For a cluster $C \subseteq F$, let M_C be the set of all minimal elements in $f^{-1}(C)$. Obviously, $M_C \neq \varnothing$.

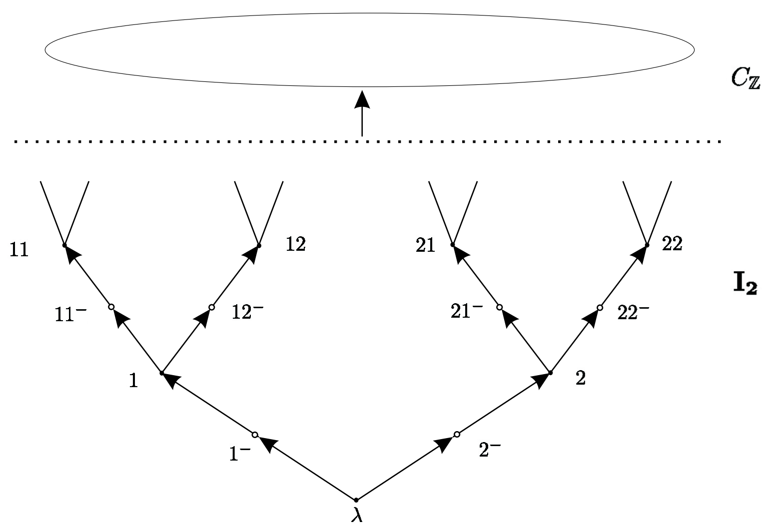

Figure 3.

Consider $w \in F$, and let $C(w)$ be its cluster. If w is not a root of G, the cluster $C(w)$ has a single $<_R$-predecessor. Choose an element $C(w)^-$ in this preceding cluster.

Let us extend f to $f' : \mathbf{I_2} \to G$ as follows: if $f(\alpha) = w,\ \alpha \neq \lambda$, we put

$$f'(\alpha^-) = \begin{cases} C(w)^-, & \text{if } \alpha \in M_{C(w)} \text{ and } C(w)^- \text{ is irreflexive,} \\ w, & \text{otherwise.} \end{cases}$$

(Informally: $f'(\alpha^-) = f(\alpha)$, whenever this is possible.)

Let us show that f' is surjective. Due to the surjectivity of f, it is sufficient to show that every irreflexive $w \in G$ is in the range of f'. So let C be the (unique) successor cluster of $w,\ x \in M_C$. Then $C^- = w$ and by definition $f'(x) = w$.

The monotonicity of f' obviously follows from the monotonicity of f and from the observation that $f'(\alpha^-)Rf(\alpha)$.

Let us prove the lift property for f'. Assuming $f'(x)Rw$, we have to find $y > x$ such that $f'(y) = w$. Consider different cases.

(1) $x \in T_2,\ w \in F$.

Then we can apply the lift property of f.

(2) $x \in T_2$, w is irreflexive.

Let C be the successor cluster of w,

$$Y := \{y \mid x \sqsubseteq y,\ y \in f^{-1}(C)\}.$$

Since f has the lift property, Y is non-empty. Choose a minimal $\alpha \in Y$. Then $x \lessdot \alpha^-$, $f'(\alpha^-) = w$.

(3) x is irreflexive, $f'(x) \in F$.

Let $x = \alpha^-$. By definition, $f'(x) = f'(\alpha)$. By the lift property of f, there exists $y \gtrdot \alpha \gtrdot x$ such that $f'(y) = w$.

(4) Both x, $f'(x)$ are irreflexive.

Let C be the successor cluster of $f'(x)$, $x = \alpha^-$. Then by definition, $f'(\alpha) = f(\alpha) \in C$. Thus $f(\alpha)Rw$ and by the lift property of f, we have $y \gtrdot \alpha \gtrdot x$ such that $f'(y) = w$.

Therefore $f' : \mathbf{I_2} \twoheadrightarrow G$, and the first assertion of Theorem follows from P-morphism Lemma and Lemma 12.

Finally, since a finite cluster C is a p-morphic image of the infinite cluster $C_{\mathbb{Z}}$, we can extend f' to a p-morphism $\mathbf{I}_\mathbf{2}^+ \twoheadrightarrow G+C$. This yields the inclusion $\mathbf{L}(\mathbf{I}_\mathbf{2}^+) \subseteq \mathbf{L}(\mathcal{K}_2) = \mathbf{L}_2$. ■

Now recall the definitions of causal and chronological future relations ($\preceq$, $\prec$) in Minkowski spacetime $\mathbb{R}^n$ (where $n \geq 2$):

$$(x_1, \ldots, x_n) \preceq (y_1, \ldots, y_n) \Leftrightarrow \sum_{i=1}^{n-1} (y_i - x_i)^2 \leq (x_n - y_n)^2 \ \&\ x_n \leq y_n,$$

$$(x_1, \ldots, x_n) \prec (y_1, \ldots, y_n) \Leftrightarrow \sum_{i=1}^{n-1} (y_i - x_i)^2 < (x_n - y_n)^2 \ \&\ x_n < y_n.$$

Let $MF_D^n = (\mathbb{R}^n, \preceq)$, $MF_C^n = (\mathbb{R}^n, \prec)$.

From [6], [12] it is known that $\mathbf{L}(MF_D^n) = \mathbf{S4.2}$ for any $n \geq 2$. Our next aim is to describe $\mathbf{L}(MF_C^n)$.

The following soundness lemma is proved easily [6]:

LEMMA 14. $\mathbf{L}(MF_C^n) \supseteq \mathbf{L}_2$ *for* $n \geq 2$.

To prove the converse, we begin with the two-dimensional case.

Let us introduce some notation. pr_1 and pr_2 denote the standard projections $\mathbb{R}^2 \longrightarrow \mathbb{R}$; $\mathrm{pr}_i(x_1, x_2) := x_i$.

Let $P, Q \in \mathbb{R}^2$, $P \neq Q$, $\mathrm{pr}_2(P) = \mathrm{pr}_2(Q)$. Take the lower isosceles right triangle[1] PRQ with the right angle at R and subtract its hypotenuse $[PQ]$. Let
$\nabla(PQ) := PRQ \backslash [PQ]$,
$K(PQ) := [RQ] \cup [RP]$.

For $m \in \mathbb{Z}$ define the points $S_m(PQ)$ on the segment $[PQ]$ as follows:
$S_0(PQ)$ is the middle of $[PQ]$,
$S_m(PQ)$ is the middle of $[PS_{m+1}(PQ)]$ for $m < 0$,
$S_m(PQ)$ is the middle of $[S_{m-1}(PQ)Q]$ for $m > 0$.

For $x \in \mathbb{R}^2$, $M \subseteq \mathbb{R}^2$ let

$$h(x) := |\mathrm{pr}_2(x)|,\ h(M) := \sup\{h(x) \mid x \in M\}.$$

Now fix the points $A := (-1, 0)$, $B := (1, 0)$, $C := (0, -1)$, and the corresponding open triangle together with the vertex C:

$$\nabla := (\nabla(AB) \backslash K(AB)) \cup \{C\}.$$

LEMMA 15. [2] *There exists a p-morphism* $(\nabla, \prec) \twoheadrightarrow \mathbf{I_2}$.

Proof. Our construction is a modification of those from [6], [12].

For $\sigma \in T_{\mathbb{Z}}$ we define points A^σ, B^σ by induction on the length $|\sigma|$. Put
$A^\lambda := A$, $B^\lambda := B$,
$A^{\sigma m} := S_m(A^\sigma B^\sigma)$, $B^{\sigma m} := S_{m+1}(A^\sigma B^\sigma)$, where $m \in \mathbb{Z}$;

$$\nabla^\sigma := \nabla(A^\sigma B^\sigma),\ K^\sigma := K(A^\sigma B^\sigma),\ W^\sigma := \bigcup_{m \in \mathbb{Z}} \nabla^{\sigma m},$$

and let C^σ be the lower vertex of ∇^σ.

One can easily check the following:

(1) $\sigma_1 \sqsubseteq \sigma_2 \Leftrightarrow \nabla^{\sigma_1} \supseteq \nabla^{\sigma_2}$.

(2) $\nabla^{\sigma_1} \cap \nabla^{\sigma_2} \neq \varnothing \Rightarrow \sigma_1 \sqsubseteq \sigma_2$ or $\sigma_2 \sqsubseteq \sigma_1$.

Note that
$h(\nabla^\lambda) = 1$,
$h(\nabla^{\sigma m}) \leq h(\nabla^{\sigma 0}) = \frac{1}{4} h(\nabla^\sigma)$, and thus

$$h(\nabla^\sigma) \leq 4^{-|\sigma|},\quad h(W^\sigma) = h(\nabla^{\sigma 0}) \leq 4^{-|\sigma|-1}.$$

Since $h(x) > 0$ for any $x \in \nabla$, there exists the longest sequence σ_x such that $x \in \nabla^{\sigma_x}$; then $x \notin W^{\sigma_x}$. Due to (2), (1), σ_x is unique. Since $x \in \nabla^{\sigma_x}$, we have

[1] All triangles are considered as closed domains on the plane.
[2] A more accurate notation for $(\nabla, \prec)$ should be $(\nabla, \prec \restriction \nabla)$.

(3) $\prec(x) \subseteq \nabla^{\sigma_x}$.

Now we define the map $f : \nabla \to I_2$ as follows.

If $x \neq C$ and $\sigma_x = m_1 \dots m_l$, $l \geq 0$ [3] , put $\tau_x := \tilde{m}_1 \dots \tilde{m}_l$, where $\tilde{m} \in \{1, 2\}$, $\tilde{m} \equiv m \pmod 2$. Let

$$f(x) := \begin{cases} \tau_x & \text{if } x \text{ is an interior point of } \nabla^{\sigma_x}; \\ \lambda & \text{if } x = C; \\ \tau_x^- & \text{otherwise.} \end{cases}$$

For $x \in K^{\sigma_x}$ put $f(x) = \tau_x^-$, otherwise $f(x) = \tau_x$ (Fig. 4).

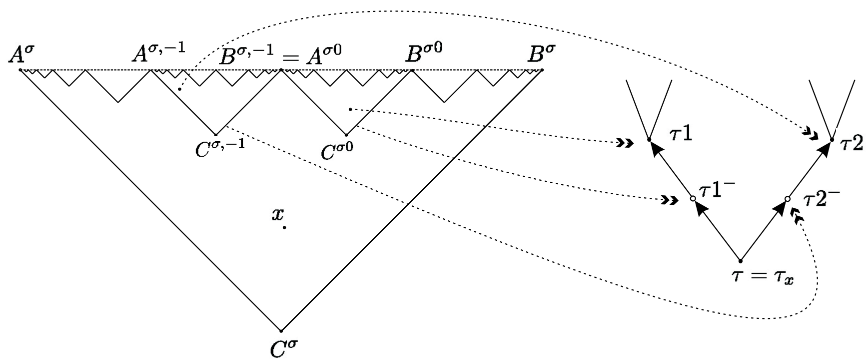

Figure 4.

Let us show that f is a p-morphism.

To check monotonicity, assume $x \prec y$. (3) implies that $y \in \nabla^{\sigma_x}$. Then $y \in \nabla^{\sigma_x} \cap \nabla^{\sigma_y}$, and so $\sigma_x \sqsubseteq \sigma_y$ or $\sigma_y \sqsubseteq \sigma_x$, by (2).

If $\sigma_y \sqsubset \sigma_x$, then $W^{\sigma_y} \supseteq \nabla^{\sigma_x}$, which contradicts $y \in \nabla^{\sigma_x}$.

If $\sigma_x \sqsubset \sigma_y$, then $\tau_x \sqsubset \tau_y$, and thus $\tau_x^- \lessdot \tau_x \lessdot \tau_y^- \lessdot \tau_y$, which implies $f(x) \lessdot f(y)$.

If $\sigma_x = \sigma_y$, then $f(x) = \tau_x$ or $f(x) = \tau_x^-$, $f(y) = \tau_x$, and again $f(x) \lessdot f(y)$.

To check the lift property for f, assume $f(x) \lessdot v$, where $v = \alpha$ or $v = \alpha^-$ for some $\alpha \in T_2$, $\sigma_x = m_1 \dots m_k$, $\alpha = i_1 \dots i_l$.

We have to find $y \succ x$ such that $f(y) = v$. Now there are two cases: $l = k$ or $l > k$.

Suppose $l = k$. Since α^- is irreflexive, we have $v = \alpha = \tau_x$.

Let $[xz_1]$ be the perpendicular to $[AB]$, $]xz_2[:=]xz_1[\setminus W^{\sigma_x}$, and let y be the middle of $]xz_2[$ (Fig. 5).

[3] If $l = 0$, $m_1 \dots m_l$ means λ.

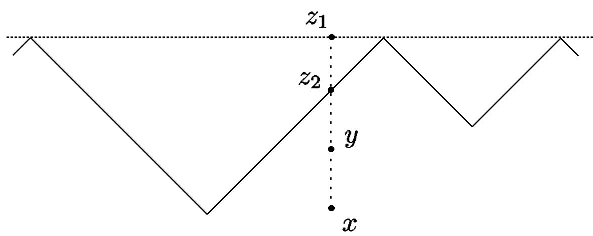

Figure 5.

Then obviously $x \prec y$. Since $y \in \nabla^{\sigma_x}$, $y \notin W^{\sigma_x}$, we obtain $\sigma_y = \sigma_x$, $f(y) = \tau_x = v$.

Now suppose $l > k$. Due to the transitivity of $\prec$ and $\lessdot$, it is sufficient to consider only the case $l = k + 1$. Then $\alpha = \tau_x i_l$.

Since $x \notin W^{\sigma_x}$, for some division point $Q = A^{\sigma_x m}$ we have $x \prec Q$ (otherwise $\prec(x) \subseteq \nabla^{\sigma_x m}$ for some m). Put

$$m' := \begin{cases} m & \text{if } m \equiv i_l (\text{mod } 2), \\ m-1 & \text{otherwise,} \end{cases}$$

and let $\rho := \sigma_x m'$. Then Q is the right or the left vertex of ∇^ρ. Since $x \prec Q$, for some $u \in K^\rho$, we have $x \prec u$.

Let $[uz_1]$ be the perpendicular to $[AB]$, $]uz_2[:=]uz_1[\setminus W^\rho$, and let y be the middle of $]uz_2[$ (Fig. 6).

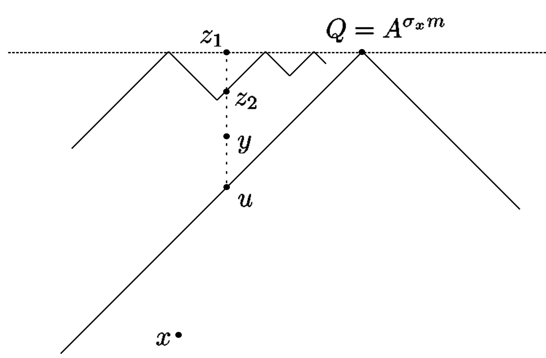

Figure 6.

Then we have $x \prec u \prec y$, $\tau_u = \tau_y = \alpha$, and thus $f(u) = \alpha^-$, $f(y) = \alpha$. Therefore $f(u) = v$ or $f(y) = v$. ∎

THEOREM 16. $\mathbf{L}(MF_C^2) = \mathbf{L}_2$.

Proof. Let $F = MF_C^2$. Since $\mathbf{L}(F) = \bigcap_{x\in\mathbb{R}^2} \mathbf{L}(F^x)$ and all cones F^x are isomorphic, we have $\mathbf{L}(F) = \mathbf{L}(F^x)$.

So consider the cone $G = F^{(0,-1)}$. In view of Theorem 13, it is sufficient to construct a p-morphism $G \twoheadrightarrow \mathbf{I_2^+} = \mathbf{I_2} + C_\mathbb{Z}$.

We extend the p-morphism $f : \nabla \twoheadrightarrow \mathbf{I_2}$ from Lemma 15 to $f' : G \twoheadrightarrow \mathbf{I_2^+}$ as follows.

$$f'(x) := \begin{cases} f(x) & \text{if } x \in \nabla, \\ c_{m(x)} & \text{if } x \notin \nabla. \end{cases}$$

Here we assume that $C_\mathbb{Z} := \{c_n \mid n \in \mathbb{Z}\}$; for $x \in G$ $m(x)$ denotes the integer part of $\mathrm{pr}_1(x)$. Then one can easily check that f' is a p-morphism; note that if $x \notin \nabla$, then $\prec(x)$ contains points y with arbitrary $m(y)$. ∎

THEOREM 17. *$L(MF_C^n) = \mathbf{L}_2$ for any $n \geq 2$.*

Proof. Similar to [6]. Let p be the projection $\mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}$:

$$p(x_1, \dots, x_{n+1}) := (x_1, x_{n+1}).$$

It is easily seen that p maps a $\prec$-cone onto a $\prec$-cone, and thus $p : MF_C^n \twoheadrightarrow MF_C^2$, whence

$$\mathbf{L}(MF_C^n) \subseteq \mathbf{L}(MF_C^2) = \mathbf{L_2}.$$

On the other hand, $\mathbf{L}(MF_C^n) \supseteq \mathbf{L_2}$ by Lemma 14. ∎

Now let us consider logics of some domains on the plane with the relation $\prec$. Let us first prove a generalization of Lemma 15.

Let PRQ be a right triangle, in which $\mathrm{pr}_1(P) < \mathrm{pr}_1(Q)$, $ang(RP) = ang(RQ) = \pi/4$, where for a straight line l, $ang(l)$ denotes the (smallest) angle between l and the x-axis.

Let F be a real-valued differentiable function whose domain contains $[\mathrm{pr}_1(P), \mathrm{pr}_1(Q)]$, and assume that

$$\forall x \in\ ]\mathrm{pr}_1(P), \mathrm{pr}_1(Q)[\ \ |F'(x)| < 1.$$

Also assume that γ is the graph of F, and $P, Q \in \gamma$. Then from Lagrange theorem we obtain that $\gamma \cap\ ]RP[\ = \gamma \cap\ ]RQ[\ = \varnothing$.

Let D be the open domain bounded by $[RP]$, $[RQ]$ and γ, and let

$$\nabla_F(PQ) := D \cup \{R\},\ \widehat{\nabla}_F := D \cup \{R\}\ \cup\ ]PR[\ \cup\ ]RQ[.$$

LEMMA 18. *There exists a p-morphism $(\nabla_F(PQ), \prec) \twoheadrightarrow \mathbf{I_2}$.*

Proof. Let $A = P$, $B = Q$, and for $\sigma \in T_{\mathbb{Z}}$ consider the same division points $A^\sigma, B^\sigma \in [PQ]$ as in the proof of Lemma 15. Take the corresponding points on the curve γ:

$$A_F^\sigma := (\mathrm{pr}_1(A^\sigma), F(\mathrm{pr}_1(A^\sigma))),\ B_F^\sigma := (\mathrm{pr}_1(B^\sigma), F(\mathrm{pr}_1(B^\sigma))),$$

and let

$$\nabla^\sigma := \widehat{\nabla}_F(A_F^\sigma B_F^\sigma),\ W^\sigma := \bigcup_{m\in\mathbb{Z}} \nabla^{\sigma m}.$$

Similarly to Lemma 15, we obtain that for every $x \in \nabla_F(PQ)$ there exists a unique sequence σ_x such that $x \in \nabla^{\sigma_x}$ and $x \notin W^{\sigma_x}$.

So we can define a map $f : \nabla_F(PQ) \longrightarrow I_2$ in the same way as in Lemma 15. Namely, for $\sigma_x = m_1 \dots m_l$, put $\tau_x := \tilde{m}_1 \dots \tilde{m}_l$. The definition of f is as follows (Fig. 7).

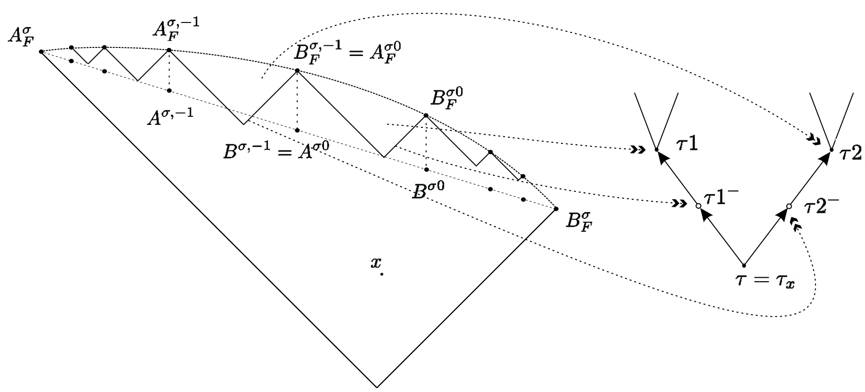

Figure 7.

$$f(x) := \begin{cases} \tau_x & \text{if } x \text{ is an interior point of } \nabla^{\sigma_x}; \\ \lambda & \text{if } x = R; \\ \tau_x^- & \text{otherwise.} \end{cases}$$

By the same argument as in Lemma 15, one can show that f is a p-morphism. ■

THEOREM 19. *Assume that X is an open convex polygon in $\mathbb{R}^2$. Then $\mathbf{L}(X, \prec)$ is either $\mathbf{L}_1$ or $\mathbf{L}_2$.*

Proof. (Cf. [12]). Let V be the highest vertex of X, and let A and B be the vertices of X that are adjacent to V.

There may be two cases:

(1) $ang(AV) < \frac{\pi}{4}$ or $ang(BV) < \frac{\pi}{4}$;

(2) $ang(AV) \geq \frac{\pi}{4}$ and $ang(BV) \geq \frac{\pi}{4}$.

Consider the first case. Assume that for example, $ang(AV) < \frac{\pi}{4}$ (Fig. 8). Take the linear function $F : \mathbb{R} \longrightarrow \mathbb{R}$ with the graph AV. Take two

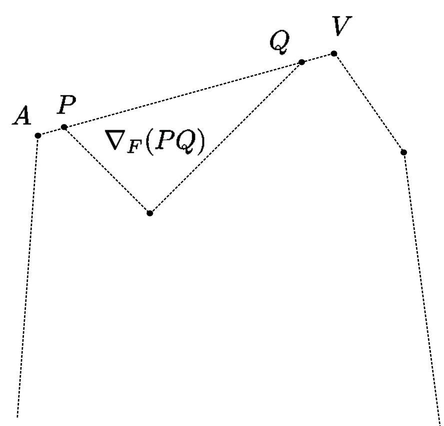

Figure 8.

points $P, Q \in [AV]$ such that $\nabla_F(PQ) \subseteq X$. Then $\nabla_F(PQ)$ is a generated subframe in $(X, \prec)$, and thus by Lemma 18,

$$\mathbf{L}(X, \prec) \subseteq \mathbf{L}(\nabla_F(PQ), \prec) \subseteq \mathbf{L}_1.$$

One can easily see that $(X, \prec)$ is serial and 2-dense, and thus we obtain $\mathbf{L}(X, \prec) = \mathbf{L}_1$.

In the case (2), take an interior point R in the triangle AVB (Fig. 9). Then there exist points $P \in [AV]$, $Q \in [BV]$ such that $ang(RP) = ang(RQ) = \frac{\pi}{4}$. The cone $Y := X \cap \prec (R)$ can be presented as $Y = \nabla(PQ) \cup D$, where $D = PQV \backslash ([PV] \cup [QV])$.

Let us show that Y can be p-morphically mapped onto $\mathbf{I_2} + C_{\mathbb{N}}$, where $C_{\mathbb{N}}$ is a countable cluster. By Lemma 18, there exists $h : \nabla(PQ) \twoheadrightarrow \mathbf{I_2}$, so let us prolong h to Y. Assume that (A_n) is a sequence of points from D converging to V. Consider the infinite sequence of natural numbers taking

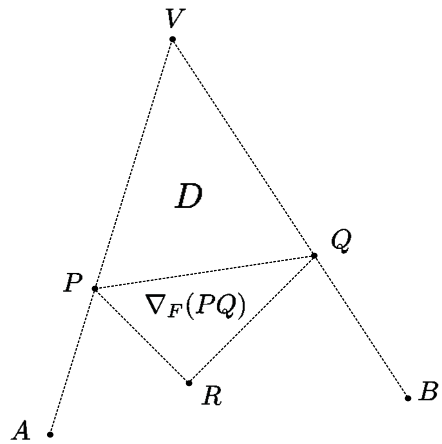

Figure 9.

every value infinitely many times, for instance $(s_n) = 1121231234\ldots$. Let $C_{\mathbb{N}} = \{c_0,\ c_1,\ c_2, \ldots\}$. Then put

$$h(A_n) := c_{s_n}.$$

For all other points $B \in D$ put $h(B) := c_0$. A straightforward argument shows that $p : Y \twoheadrightarrow \mathbf{I_2} + C_{\mathbb{N}}$. Therefore,

$$\mathbf{L}(X, \prec) \subseteq \mathbf{L}(Y) \subseteq \mathbf{L}_2.$$

Since X is convex, the condition (2) implies that the frame $(X, \prec)$ is confluent, and thus $\mathbf{L}(X, \prec) \supseteq \mathbf{L}_2$. ∎

THEOREM 20. *Let X be an open connected domain in $\mathbb{R}^2$ bounded by a closed smooth curve. Then $\mathbf{L}(X, \prec) = \mathbf{L}_1$.*

Proof. Let γ be the boundary of X. Due to smoothness, it can be presented as $\{(x(t), y(t)) \mid t \in [0,1]\}$, where $x(t)$ and $y(t)$ are smooth functions on $[0,1]$ such that for any t, $(\dot{x}(t), \dot{y}(t)) \neq (0,0)$. Let $C = (x(t_0), y(t_0))$ be the highest point of γ (which may be not unique). Then $\dot{y}(t_0) = 0,\ \dot{x}(t_0) \neq 0$, and without any loss of generality, we may assume that $\dot{x}(t_0) > 0,\ t_0 \neq 0,\ t_0 \neq 1$. Since the derivatives $\dot{x}$ and $\dot{y}$ are continuous, there exists an interval $]t_1, t_2[\ni t_0$, where $\dot{x}(t) > \varepsilon,\ |\dot{y}(t)| < \varepsilon$ for some $\varepsilon > 0$. Then $x(t)$ is invertible in this interval, and the corresponding part of γ is a graph of some function $F :]x_1, x_2[\longrightarrow \mathbb{R}$, where $x_1 = x(t_1),\ x_2 = x(t_2)$. Now it follows that $|F'(x)| = |\dot{y}(t)/\dot{x}(t)| < 1$ for $x \in]x_1, x_2[$.

We can choose points P, Q in this part of γ such that $\nabla_F(PQ) \subseteq X$. Since $\nabla_F(PQ)$ is a cone in $(X, \prec)$ and by Lemma 18, we have

$$\mathbf{L}(X, \prec) \subseteq \mathbf{L}(\nabla_F(PQ), \prec) \subseteq \mathbf{L}_1.$$

It remains to note that $\mathbf{L}(X, \prec)$ is 2-dense. In fact, for any $x \in X$ there exists an open disk $U \subseteq X$ containing x. Then every two points $y,\ z \in \prec(x)$ are $\prec$-accessible from some $u \in U \cap \prec(x)$.

Thus $\mathbf{L}(X, \prec) \supseteq \mathbf{L}_1$. ■

5 Conclusion

Logics of relativistic time studied so far are examples of many-dimensional modal logics that are rather simple from the computational viewpoint. To say more on that topic, let us recall the connection between relativistic modal logics and modal products [13].

Consider the product frame
$(\mathbb{R}, <)^2 := (\mathbb{R}, <) \times (\mathbb{R}, <) := (\mathbb{R}^2, R_1, R_2)$,
where
$(x, y) R_1 (x', y') \Leftrightarrow x < x' \ \&\ y = y'$;
$(x, y) R_2 (x', y') \Leftrightarrow x = x' \ \&\ y < y'$.

The logic of $\mathbf{L}((\mathbb{R}, <)^2)$ is known to be Π^1_1-complete, and therefore it is not recursively axiomatizable [11], [4]. But our $\mathbf{L}_2$ is its rather natural decidable fragment. In fact, the frames $(\mathbb{R}^2, R_1 \circ R_2)$ and $(\mathbb{R}^2, \prec)$ are isomorphic; an isomorphism is given by rotation. Thus the corresponding logics can be identified by translation φ from 1-modal to 2-modal formulas reading $\Box$ as $\Box_1\Box_2$:

COROLLARY 21. $\mathbf{L}_2 = \{A \mid (\mathbb{R}, <)^2 \vDash \varphi(A)\}$.

The proofs in Section 4 easily transfer to the rational case, and in the same way we obtain:

COROLLARY 22. $\mathbf{L}_2 = \{A \mid (\mathbb{Q}, <)^2 \vDash \varphi(A)\}$.

Note that the whole logic $\mathbf{L}((\mathbb{Q}, <)^2)$ is undecidable [11], [4], but unlike the real case, it is recursively enumerable and coincides with the corresponding product of modal logics $\mathbf{L}(\mathbb{Q}, <) \times \mathbf{L}(\mathbb{Q}, <)$ [13], [4].

It remains a serious open problem, whether all "φ-fragments" of products of linear modal logics are decidable. Probably, the most interesting unknown case is $\mathbf{L}(\mathbb{Z}, <) \times \mathbf{L}(\mathbb{Z}, <)$. A related question [6] is about properties of the logic of discrete Minkowski spacetime (which coincides with $\mathbf{L}((\mathbb{Z}, <)^2)$).

The above two corollaries can be reformulated in terms of classical first-order theories in the style of [5]. Viz., consider the first-order language $\mathcal{L}$ with binary predicates $<,\ P_1,\ P_2, \ldots$ Every 1-modal propositional formula

A translates into an $\mathcal{L}$-formula $A^*(x,y)$:
$p_n^*(x,y) := P_n(x,y)$,
$\bot^* := \bot$,
$(A \to B)^* := A^* \to B^*$,
$\Diamond A^*(x,y) := \exists x_1 \exists y_1 \ (x < x_1 \wedge y < y_1 \wedge A^*(x_1,y_1))$.

Let $Th^2(W,<)$ be the $\mathcal{L}$-theory of all structures $(W,<,\dots)$ with fixed $(W,<)$ and varying interpretations of $P_1,\ P_2,\dots$ Then we obtain

COROLLARY 23. $\mathbf{L}_2 = \{A \mid Th^2(\mathbb{R},<) \vdash A^*\} = \{A \mid Th^2(\mathbb{Q},<) \vdash A^*\}$.

One can show that the same logics arise in the many-dimensional case. In fact, consider the n-dimensional product $(\mathbb{R},<)^n = (\mathbb{R}^n, R_1,\dots,R_n)$, where $(x_1,\dots,x_n)R_i(y_1,\dots,y_n)$ iff $x_i < y_i$ & $\forall j \neq i \ x_j = y_j$. Then a standard projection is a p-morphism $p : (\mathbb{R},<)^n \twoheadrightarrow (\mathbb{R},<)^2$.

Let φ_n be translation from 1-modal to n-modal formulas interpreting $\Box$ as $\Box_1 \dots \Box_n$. Then similarly to Corollaries 21, 22, we have

COROLLARY 24. $\mathbf{L}_2 = \{A \mid (\mathbb{R},<)^n \vDash \varphi_n(A)\} = \{A \mid (\mathbb{Q},<)^n \vDash \varphi_n(A)\}$.

This Corollary also has a classical analogue similar to Corollary 23; we leave the precise details for the reader.

The logics $\mathbf{L}_1, \mathbf{L}_2$ can also be interpreted as fragments of interval temporal logics. Logics of intervals have motivations in Computer Science, Linguistics, and Philosophy; the reader is addressed to [9] for further references and a brief overview of this area. Let us recall that there are 13 basic relations between intervals in a linearly ordered set, and the corresponding full modal logic of these accessibility relations is undecidable, according to the result by Halpern and Shoham [8]. This happens in most cases, in particular, for integers, rationals, and reals.

On the other hand, for rational and real time, as stated in [13], some natural fragments of interval logics are equivalent to the well-known systems **S4**, **S4.2**. A similar property holds for $\mathbf{L}_1,\ \mathbf{L}_2$. In fact, let $I(W,<)$ be the set of all nontrivial closed intervals (segments) in a linearly ordered set $(W,<)$. Obviously, we can identify $I(W,<)$ with the half-plane $\{(x,y) \in W^2 \mid x < y\}$.

Consider the following relations between intervals.

$$(x_1,y_1) \sqsubset (x_2,y_2) := x_2 < x_1 \ \& \ y_1 < y_2 \ (\text{“during”});$$

$$(x_1,y_1) < (x_2,y_2) := x_1 < x_2 \ \& \ y_1 < y_2 \ (\text{“weakly earlier”}),$$

and their converses $\sqsupset, >$. (Note that $<$ is not among the basic 13 relations, because $>$ is the union of “later than” and “overlaps”.) Fig. 10 shows the points of $I(W,<)$ accessible from a certain point by these relations. So we can see that on the real plane the cone $\sqsubset$ is isomorphic to the triangle

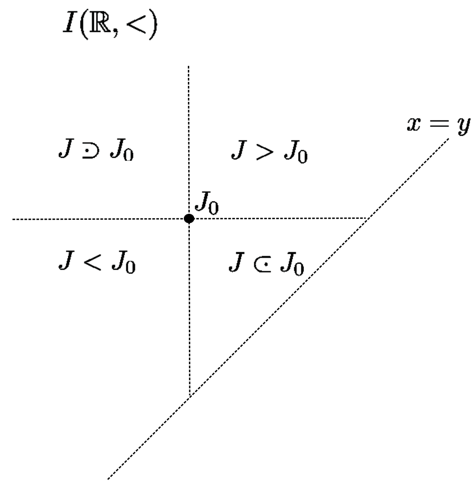

Figure 10.

$(\nabla, \prec)$ considered in Section 3, and thus its logic is $\mathbf{L}_1$. Three other cones are confluent, and their logic is $\mathbf{L}_2$. The case of rationals is completely analogous. So we obtain

COROLLARY 25. *Let* $P = (\mathbb{R}, <)$ *or* $(\mathbb{Q}, <)$. *Then*
$\mathbf{L}(I(P),\ \ni) = \mathbf{L}_1$,
$\mathbf{L}(I(P), <) = \mathbf{L}(I(P), >) = \mathbf{L}(I(P), \in\ ) = \mathbf{L}_2$.

Finally let us make some remarks on the computational complexity of $\mathbf{L_1}$, $\mathbf{L_2}$. It is well-known that their reflexive analogues $\mathbf{S4}$, $\mathbf{S4.2}$ are PSPACE-complete [2]. Our proof of the f.m.p. in Section 3 motivates the same complexity bound for $\mathbf{L_1}$, $\mathbf{L_2}$. The corresponding result recently proved by the first author will be published in the sequel.

Also note that there probably exists an alternative proof of the f.m.p. based the method from [2] (Theorem 11.52). However this method does not provide a good complexity bound.

There remain many open problems in the field. For example, nothing is known about decidability, complexity and expressive power of relativistic polymodal logics. The same questions certainly make sense in the context of general relativity theory (a brief discussion can be found in [1]) or for many-dimensional analogues of interval logics. So there is enough room for further investigations of spacetime logics.

Acknowledgements

The authors thank the anonymous referees for very useful and detailed comments on the earlier versions of the paper.

The work on this paper was partly supported by Russian Foundation for Basic Research and ESPRC project on many-dimensional modal logic.

BIBLIOGRAPHY

[1] J. Burgess. Logic and time. Journal of Symbolic Logic, v. 44(1979), 566-582.
[2] A. Chagrov, M. Zakharyaschev. Modal logic. Oxford University Press, 1997.
[3] K. Fine. Logics containing K4. I. Journal of Symbolic Logic, v. 39(1974), 31-42.
[4] D. Gabbay, A. Kurusz, F. Wolter, M. Zakharyaschev. Many-dimensional modal logics: theory and applications. Studies in Logic, Elsevier, 2003.
[5] D. Gabbay, V. Shehtman. Products of modal logics. II. Relativised quantifiers in classical logic. Logic Journal of the IGPL, v. 8 (2000), No. 2, p. 165-210.
[6] R. Goldblatt. Diodorean modality in Minkowski spacetime. Studia Logica, v. 39 (1980), 219-236.
[7] R. Goldblatt. Logic of time and computation. CSLI Lecture Notes, No. 7. Stanford, 1987.
[8] J. Halpern, Y. Shoham. A propositional modal logic of time intervals. Journal of the ACM, v. 38(1991), 935-962.
[9] M. Marx, Y. Venema. Multi-dimensional modal logic. Applied Logic Series, v.4. Kluwer Academic Publishers, 1997.
[10] A.N. Prior. Past, present and future. Clarendon Press, Oxford, 1967.
[11] M. Reynolds, M. Zakharyaschev. On the products of linear modal logics. Journal of Logic and Computation, v. 11 (2001), 909-931.
[12] V.B. Shehtman. Modal logics of domains on the real plane. Studia Logica, v. 42 (1983), 63-80.
[13] V.B. Shehtman. On some two-dimensional modal logics. In: 8th International Congress on Logic, Methodology, and Philosophy of Science, Moscow, 1987. Abstracts, v.1, 326-330, 1987.

Ilya Shapirovsky
Department of mathematical logic
Faculty of mathematics and mechanics
Moscow State University
Moscow, Russia
E-mail: ilshapir@lpcs.math.msu.ru

Valentin Shehtman
Institute of Infomation Transmission Problems
B. Karetny 19
101447 Moscow, Russia
E-mail: shehtman@lpcs.math.msu.ru