Every world can see a Sahlqvist world

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abstract. We introduce a class of propositional modal logics axiomatized by infinite sequences of formulas in special form. Two logics of this type are known from [10] and [13]. Although the axioms are beyond Sahlqvist class and its generalization defined in [8], all the resulting logics are still complete with respect to elementary classes of frames. For two particular examples of these logics related to the McKinsey axiom, we also study elementarity and finite model property.

1 Introduction

The starting point for our research is the well-known Sahlqvist Theorem. For about thirty years this result was considered as the strongest one giving a syntactic sufficient condition for completeness and first-order definability (elementarity) in modal logic. More recent studies show that Sahlqvist class can be extended to a larger class of "inductive" modal formulas inheriting both completeness and elementarity [8].

Now there is natural question: what happens beyond this new class? It is well-known that completeness or definability may be lost. Perhaps the simplest counterexample is given by the McKinsey axiom $\Box\Diamond p \to \Diamond\Box p$, which is non-elementary, but still complete and even has the finite model property. So after a slight variation of Sahlqvist formulas we may still hope to preserve at least some of nice properties.

On the other hand, recently Ian Hodkinson has found a precise description of quasi-elementary logics (i.e. those complete with respect to elementary classes of Kripke frames) [9]: he defines a translation from a first-order theory T into a set of hybrid modal formulas, and next – into a set of pure modal formulas axiomatizing exactly the class of models of T. Moreover, the corresponding hybrid formulas are identified as "quasipositive". This general result is quite impressive, but the suggested axiomatization method may be not optimal in particular cases.

The class of modal logics studied in this paper is certainly covered by Hodkinson's theorem, but we propose a simpler description, which does not directly follow from [9]. Namely, consider different versions of a modal formula $\varphi(p_1, \dots, p_k)$ with disjoint sets of proposition letters:

$$\varphi' = \varphi(p_{k+1}, \dots, p_{2k}),\ \varphi'' = \varphi(p_{2k+1}, \dots, p_{3k}), \dots$$

and put

$$\varphi^n_\Diamond = \Diamond(\varphi \wedge \varphi' \wedge \dots \wedge \varphi^{(n-1)}),$$

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$$L_\Diamond(\varphi) = \mathbf{K} + \{\varphi^n_\Diamond \mid n \geq 0\}.$$

As we show, every logic $L_\Diamond(\varphi)$ is quasi-elementary, whenever φ is a Sahlqvist formula (or even an inductive formula). By Fine – Van Benthem Theorem (cf. [4, Theorem 10.19]) this implies canonicity. An appropriate first-order condition can be obtained in a standard way:

$$\forall x \exists t (xRt \wedge \Phi(t)),$$

where Φ is the first-order correspondent of φ, i.e. "every world sees a φ-world". However this condition does not always characterize the frames for $L_\Diamond(\varphi)$ — in general we cannot state that this logic is elementary, and it may be non-axiomatizable by inductive formulas. On the other hand, we show that it is finitely elementary, i.e. all its finite frames satisfy the above mentioned condition.

Also note that $L_\Diamond(\varphi)$ is the union of the increasing sequence of logics $L^n_\Diamond(\varphi) = \mathbf{K} + \varphi^n_\Diamond$, so it may be not finitely axiomatizable.

Two examples of such logics have been known from the literature, both are non-elementary and non-finitely axiomatizable.

The first logic $L_\Diamond(\Box p \to p)$ was introduced and studied by Hughes [10], who also proved the finite model property and decidability.

The second example is Lemmon's logic $KM^\infty = L_\Diamond(\Diamond p \to \Box p)$ [13]. This logic is even worse in some respect — recent results show that although KM^∞ is canonical itself, it cannot be axiomatized by canonical formulas and all the logics $\mathbf{K} + \varphi^n_\Diamond$ are not canonical (for $\varphi = \Diamond p \to \Box p$) [7]. Note that $\varphi^1_\Diamond = \Diamond(\Diamond p_1 \to \Box p_1)$ is an equivalent form of the McKinsey axiom, for which the non-canonicity was obtained earlier [6]. To add to the list of negative results, we show that any logic between $\mathbf{K} + \varphi^1_\Diamond$ and KM^∞ is non-elementary. On the other hand, KM^∞ has the fmp, by Fine's Theorem on uniform logics, cf. [4]. This result can be improved and in fact KM^∞ is PSPACE-complete, as we are going to show in the second part of this paper.

In this paper we also consider a third example, viz. $L^{fw}_\omega = L_\Diamond(p \to \Box p)$. This logic happens to be "more regular": it is canonical, elementary and modally defines "McKinsey property" (which is a first-order equivalent of the McKinsey axiom in the transitive case, cf. [2]; moreover, L^{fw}_ω is finitely axiomatizable, has the fmp and therefore is decidable (and as we shall prove in the sequel, PSPACE-complete)

Note that all our logics are axiomatized by modal formulas of the form $\Diamond(\varphi_1 \wedge \cdots \wedge \varphi_n)$, where φ_i are "good" (say, Sahlqvist). The use of only $\Diamond$ and $\wedge$ seems "relatively safe" in this context, and so there is a hope to extend our results to a larger class of formulas. In the final section we briefly discuss some further results and related open problems.

2 Basic concepts

We assume the reader is familiar with tools and techniques in modal logic such as the canonical model construction and the filtration method. We also assume the reader is at home with well-known results such as Sahlqvist's theorem and Ladner's theorem. For more on these see Blackburn, de Rijke and Venema [3], Chagrov and Zakharyaschev [4] and Kracht [11]. Still in this Section we recall some basic notions, for the sake of terminology and notation.

2.1 Syntax

Let $PV = \{p_1,\ p_2, \dots\}$ be a countable set of proposition letters, with typical members denoted by p, q, etc. Modal formulas over PV are built using the constant $\bot$, the unary connective $\Box$ and the binary connective $\to$. Other constructs are defined as usual, in particular $\Diamond\varphi$ is the abbreviation for $\neg\Box\neg\varphi$.

For a formula φ, $cl(\varphi)$ denotes the set of all subformulas of φ. Put $PV(\varphi) = cl(\varphi) \cap PV$. The notation $\varphi(p_1, \dots, p_n)$ means $\{p_1, .., p_n\} \supseteq PV(\varphi)$. For formulas $\psi_1, \dots, \psi_n$, $\varphi(\psi_1, \dots, \psi_n)$ denotes the result of simultaneous substitution of $\psi_1, \dots, \psi_n$ for $p_1, \dots, p_n$ in φ. For a set x of formulas, let $x^{\Box} = \{\varphi \mid \Box\varphi \in x\}$.

A *(normal) modal logic* is a set L of formulas that contains all tautologies, the formula $\Box(p \to q) \to (\Box p \to \Box q)$ and that is closed under the standard rules: Modus ponens, Uniform substitution, and Generalization (given φ infer $\Box\varphi$).

For a set Γ of formulas, $L + \Gamma$ denotes the smallest modal logic containing $(L \cup \Gamma)$.

A formula φ is said to be *L-deducible* from a set Γ of formulas, in symbols $\Gamma \vdash_L \varphi$ if there exists formulas $\varphi_1, \dots, \varphi_n \in \Gamma$ such that $(\varphi_1 \wedge \dots \wedge \varphi_n \to \varphi) \in L$. A set Γ of formulas is called *L-consistent* if $\Gamma \nvdash_L \bot$.

Recall the inductive definition of *uniform modal formulas*. Any formula without modal operators is a uniform formula of degree 0; uniform formulas of degree $n+1$ are built from the set

$$\{\Box\psi \mid \psi \text{ is a uniform formula of degree } n\}$$

using boolean connectives.

A logic is called *uniform* if it can be axiomatized by uniform formulas.

2.2 Semantics

As usual, a *(Kripke) frame* is a pair $\mathcal{F} = (W, R)$, where W is a non-empty set of worlds and R is a binary relation on W. A world $x \in W$ is called *final* in $\mathcal{F}$ if $R(x) \subseteq \{x\}$ and *deterministic* in $\mathcal{F}$ if $card(R(x)) \leq 1$. $\mathcal{F}$ is called *serial* if for all worlds x in W, $R(x) \neq \varnothing$.

For a world x in W and sets $W_1, \dots, W_n \subseteq W$, we say that $\{W_1, \dots, W_n\}$ is a *cover* of $R(x)$ if $R(x) \subseteq W_1 \cup \dots \cup W_n$.

A *(Kripke) model* based on $\mathcal{F}$ is a pair $\mathcal{M} = (\mathcal{F}, V)$, where V is a function assigning to each proposition letter p a subset $V(p)$ of W. The inductive definition of the truth value of a formula φ at a world x in a model $\mathcal{M}$ is standard. $\mathcal{M}, x \vDash \varphi$ denotes that φ is true at x in $\mathcal{M}$.

A formula φ is called *true* (respectively *satisfiable*) in a model $\mathcal{M} = (W, R, V)$, in symbols $\mathcal{M} \vDash \varphi$, if φ is true at any (resp., some) world in W; φ is *valid* in a frame $\mathcal{F}$, in symbols $\mathcal{F} \vDash \varphi$, if φ is true in all models based on $\mathcal{F}$. φ is *valid* at $\mathcal{F}, x$ (notation: $\mathcal{F}, x \vDash \varphi$) if it is true at x in all models based on $\mathcal{F}$. A formula φ is said to be valid in a class $\mathcal{C}$ of frames, in symbols $\mathcal{C} \vDash \varphi$, if φ is valid in all frames in $\mathcal{C}$. We say that a set L of formulas is valid in a frame $\mathcal{F}$, in symbols $\mathcal{F} \vDash L$, if all formulas from L are valid in $\mathcal{F}$.

Every frame can also be regarded as a first-order structure, and we use the same sign $\vDash$ to denote the truth of a first-order formula in this structure.

The *modal logic of a class of frames* $\mathcal{C}$ is defined as $\mathbf{L}(\mathcal{C}) = \{\varphi \mid \mathcal{C} \vDash \varphi\}$. A logic L is called *complete* with respect to $\mathcal{C}$ if $L = \mathbf{L}(\mathcal{C})$. L is said to have the *finite model property* (fmp) if it is complete with respect to a class of finite frames.

We say that a set of modal formulas Γ *modally defines* the class of frames $\mathbf{Fr}(\Gamma) := \{\mathcal{F} \mid \mathcal{F} \vDash \Gamma\}$; Γ *modally defines* $\mathcal{C}$ *within a class* $\mathcal{C}'$ if $\mathcal{C} = \mathbf{Fr}(\Gamma) \cap \mathcal{C}'$. We say that a formula φ modally defines $\mathcal{C}$ [within $\mathcal{C}'$] if the set $\{\varphi\}$ does.

A modal logic L is called *strongly complete* with respect to a class $\mathcal{C}$ of frames if for any L-consistent set Γ of formulas, there exists a model $\mathcal{M}$ based on a frame from $\mathcal{C}$ such that all formulas from Γ are simultaneously true at some world in $\mathcal{M}$.

Recall that a formula φ is *locally elementary* if for some first-order formula Φ with only one free variable t we have: for any frame $\mathcal{F}$ and any world a in $\mathcal{F}$, $\mathcal{F}, a \vDash \varphi$ iff $\mathcal{F} \vDash \Phi(a)$. In this case Φ is called a *local first-order correspondent* of φ.

A set Γ of modal formulas is called *elementary* (respectively, Δ-*elementary*) if the class $\mathbf{Fr}(\Gamma)$ is elementary (respectively, Δ-elementary), i.e. if $\mathbf{Fr}(\Gamma)$ is the class of models of some first-order formula (resp., theory). Every locally elementary formula is obviously elementary.

A modal logic of the form $\mathbf{L}(\mathcal{C})$, for an elementary $\mathcal{C}$, is called *quasi-elementary* (or *elementarily generated*).

3 Every world can see a φ-world

In this section we describe a family of quasi-elementary and finitely elementary logics.

For a tuple of proposition letters $\mathbf{p} = (p_1, \ldots, p_k)$ let

$$\mathbf{p}^n := (p_{kn+1}, \ldots, p_{kn+k})$$

for $n \geq 0$. So $\mathbf{p}^0 = \mathbf{p}$ and all the tuples $\mathbf{p}^n$ are disjoint.

For a modal formula $\varphi(\mathbf{p})$ put

$$\varphi^n_\Diamond := \Diamond(\varphi(\mathbf{p}^0) \wedge \cdots \wedge \varphi(\mathbf{p}^{n-1}))$$

(in particular, $\varphi^0_\Diamond := \Diamond\top$), and also

$$L_\Diamond(\varphi) := \mathbf{K} + \{\varphi^n_\Diamond \mid n \geq 0\},$$

$$L^n_\Diamond(\varphi) := \mathbf{K} + \varphi^n_\Diamond.$$

The following is rather trivial.

PROPOSITION 1. $L^0_\Diamond(\varphi) \subseteq L^1_\Diamond(\varphi) \subseteq L^2_\Diamond(\varphi) \ldots \subseteq L_\Diamond(\varphi)$.

Now let φ be a locally elementary formula, and let $\Phi(t)$ be its local first-order correspondent. Consider the class of frames

$$\mathcal{C}_\Diamond(\varphi) := \{\mathcal{F} \mid \mathcal{F} \vDash \forall x \exists t (xRt \wedge \Phi(t))\}.$$

LEMMA 2. *For a locally elementary formula φ, $\mathcal{C}_\Diamond(\varphi) \subseteq \mathbf{Fr}(L_\Diamond(\varphi))$.*

Proof. Suppose $\mathcal{F} \in \mathcal{C}_\Diamond(\varphi(\mathbf{p}))$. This means that for any $a \in \mathcal{F}$ there exists $b \in R(a)$ such that $\mathcal{F} \vDash \Phi(b)$, which is equivalent to $\mathcal{F}, b \vDash \varphi$, by the definition of Φ. But the latter implies $\mathcal{F}, b \vDash \varphi(\mathbf{p}^n)$ for any n, hence $\mathcal{F}, a \vDash \varphi^n_\Diamond$, and thus $\mathcal{F}, a \vDash L_\Diamond(\varphi)$. ■

A formula φ is *locally d-persistent* if for any descriptive (general) frame $(\mathcal{F}, \mathcal{D})$ and any world a, $(\mathcal{F}, \mathcal{D}), a \vDash \varphi$ implies $\mathcal{F}, a \vDash \varphi$ (the notion of a *descriptive frame* is defined in a standard way, see e.g. [4]).

THEOREM 3. *Let φ be a locally elementary and locally d-persistent modal formula. Then*

1. *the canonical frame for $L_\Diamond(\varphi)$ is in $\mathcal{C}_\Diamond(\varphi)$;*
2. *$L_\Diamond(\varphi)$ is canonical and therefore strongly complete with respect to $\mathcal{C}_\Diamond(\varphi)$.*

Proof. Let us prove (1); then (2) readily follows from Lemma 2 and the properties of the canonical model.

Let $\mathcal{F} = (W, R)$ be the canonical frame for $L_\Diamond(\varphi(\mathbf{p}))$, $\mathbf{p} = (p_1, \ldots, p_k)$. For any $a \in W$, put

$$a^+ := a^\Box \cup \{\varphi(\psi_1, \ldots, \psi_k) \mid \psi_1, \ldots, \psi_k \text{ are arbitrary modal formulas}\}.$$

Claim 1 a^+ is $L^\Diamond(\varphi)$-consistent.

Suppose the contrary. Then for some formulas $\gamma_1, \ldots, \gamma_m \in a^\Box$ and for some k-tuples of formulas $\overline{\psi}_1, \ldots, \overline{\psi}_n$ we have $L_\Diamond(\varphi) \vdash \gamma_1 \wedge \cdots \wedge \gamma_m \wedge \varphi(\overline{\psi}_1) \wedge \cdots \wedge \varphi(\overline{\psi}_n) \to \bot$.

Since $\Diamond(\varphi(\overline{\psi}_1) \wedge \cdots \wedge \varphi(\overline{\psi}_n))$ is a substitution instance of $\varphi^n_\Diamond$, we have $\Diamond(\varphi(\overline{\psi}_1) \wedge \cdots \wedge \varphi(\overline{\psi}_n)) \in a$, so for some $b \in R(a)$ we have $\varphi(\overline{\psi}_1), \ldots, \varphi(\overline{\psi}_n) \in b$. Since every γ_i is in $a^\Box$, we also have $\gamma_1, \ldots, \gamma_m \in b$, so $\bot \in b$, which is a contradiction. Q.e.d.

By the Lindenbaum Lemma, there exists $b \in W$ such that $a^+ \subseteq b$. So $a^\Box \subseteq b$, thus aRb.

Now consider the general canonical frame $(\mathcal{F}, \mathcal{D})$ for $L_\Diamond(\varphi)$; recall that

$$\mathcal{D} = \{V_0(\psi) \mid \psi \text{ is a modal formula}\},$$

where V_0 is the canonical valuation.

CLAIM 2 $(\mathcal{F}, \mathcal{D}), b \vDash \varphi$.

In fact, for a valuation V in $(\mathcal{F}, \mathcal{D})$ let us show that $b \in V(\varphi)$. By definition of $\mathcal{D}$, for every i there exists a modal formula ψ_i such that $V(p_i) = V_0(\psi_i)$; let $\overline{\psi} = (\psi_1, \ldots, \psi_k)$. Then by induction it follows that

$$V(\varphi(\mathbf{p})) = V_0(\varphi(\overline{\psi})).$$

But $b \supseteq a^+$, so $\varphi(\overline{\psi}) \in b$, and thus $b \in V_0(\varphi(\overline{\psi})) = V(\varphi)$. Q.e.d.

Since $(\mathcal{F}, \mathcal{D})$ is descriptive and φ is locally d-persistent, from Claim 2 we obtain $\mathcal{F}, b \vDash \varphi$, i.e. $\mathcal{F} \vDash \Phi(b)$; thus $\mathcal{F} \vDash \forall x \exists t (xRt \wedge \Phi(t))$. ■

For a set Γ of modal formulas we can also define

$$L_\diamond(\Gamma) := \mathbf{K} + \{\varphi_\diamond^n \mid \varphi \in \Gamma,\ n \geq 0\},$$

$$\mathcal{C}_\diamond(\Gamma) := \bigcap_{\varphi \in \Gamma} \mathcal{C}_\diamond(\varphi).$$

So we have

COROLLARY 4. *If Γ is a set of locally elementary and locally d-persistent modal formulas, then $L_\diamond(\Gamma) = \mathbf{L}(\mathcal{C}_\diamond(\Gamma))$.*

COROLLARY 5. *If Γ is a set of inductive modal formulas, then $L_\diamond(\Gamma) = \mathbf{L}(\mathcal{C}_\diamond(\Gamma))$, and thus this logic is canonical and strongly complete with respect to $\mathcal{C}_\diamond(\Gamma)$.*

Proof. In fact, every inductive formula is locally elementary and locally d-persistent [8]. ■

DEFINITION 6. A modal logic L is called *finitely elementary* if there exists a first-order formula Φ such that for any finite frame $\mathcal{F}$, $\mathcal{F} \vDash L$ iff $\mathcal{F} \vDash \Phi$.

THEOREM 7. *If φ is a locally elementary modal formula, then $L_\diamond(\varphi)$ is finitely elementary.*

Proof. For a formula $\varphi(\mathbf{p})$, $\mathbf{p} = (p_1, \ldots, p_k)$ with a local first-order correspondent Φ, let us prove that $\mathcal{F} \vDash L_\diamond(\varphi)$ iff $\mathcal{F} \in \mathcal{C}_\diamond(\varphi)$ for any finite frame $\mathcal{F}$. The direction "if" is already proved in Lemma 2.

So consider a finite frame $\mathcal{F} = (W, R)$ with $card(W) = n$, such that $\mathcal{F} \vDash L_\diamond(\varphi)$. We have to show that $\mathcal{F} \vDash \forall x \exists t (xRt \wedge \Phi(t))$.

Consider the following Kripke model $\mathcal{M} = (\mathcal{F}, V)$. Take the N-element set $\mathcal{P}(W)^k$ of all k-tuples of subsets of W (where $N = 2^{nk}$) and put it in some order: $\mathbf{W}^0, \mathbf{W}^1, \ldots, \mathbf{W}^{N-1}$. If $\mathbf{W}^j = (W_1^j, \ldots, W_k^j)$, we define

$$V(p_{jk+i}) := W_i^j.$$

So we have defined $V(p_m)$ for $m = 1, \ldots, Nk$, and we assume that $V(p_m)$ is arbitrary for all other m.

Now take any $a \in W$. Since $\mathcal{M}, a \vDash \varphi_\Diamond^N$, there exists $b \in R(a)$ such that $\mathcal{M}, b \vDash \varphi(\mathbf{p}^0) \wedge \dots \wedge \varphi(\mathbf{p}^{N-1})$. Then we claim that $\mathcal{F}, b \vDash \varphi$. In fact, consider an arbitrary valuation V' in $\mathcal{F}$. By our construction, there exists j such that $\mathbf{W}^j = (V'(p_1), \dots, V'(p_k))$, i.e.

$$V'(p_i) = V(p_{jk+i})$$

whenever $1 \le i \le k$. Hence by induction it easily follows that

$$V'(\varphi(\mathbf{p})) = V(\varphi(\mathbf{p}^j)).$$

Now since $\mathcal{M}, b \vDash \varphi(\mathbf{p}^j)$, we obtain $(\mathcal{F}, V'), b \vDash \varphi$. Thus $\mathcal{F}, b \vDash \varphi$, which is equivalent to $\mathcal{F} \vDash \Phi(b)$. Eventually $\mathcal{F} \vDash \forall x \exists y (xRy \wedge \Phi(y))$. ■

The following simple fact is motivated by Proposition 6.2 from [9], though it is not formulated explicitly in that paper.

PROPOSITION 8. *If a recursively axiomatizable and finitely elementary modal logic has the fmp, then it is decidable.*

Proof. In fact, if finite frames for L are defined by a certain first-order sentence, then the set of finite L-frames (whose worlds are identified with integers) is decidable. Together with the fmp, this implies the co-enumerability of L. So since L is recursively enumerable, it is decidable. ■

Note that in general a recursively axiomatizable logic with the fmp can be undecidable (the three-dimensional logic $\mathbf{K}^3$ is a typical example), but this does not affect the logics considered in the present paper.

4 Every world can see a final world

4.1 Definitions

In this section we consider a particular case of the above construction, when $\varphi = (p_1 \to \Box p_1)$. The corresponding first-order formula is $\Phi(t) = \forall x (tRx \to t = x)$.

Let

$$\alpha_n := \varphi_\Diamond^n = \Diamond((p_1 \to \Box p_1) \wedge \ldots \wedge (p_n \to \Box p_n)).$$

Consider the modal logics (for $0 \le n < \omega$):

$$L_n^{fw} := \mathbf{K} + \alpha_n, \; L_\omega^{fw} := L_\Diamond(\varphi) = \mathbf{K} + \{\alpha_0, \; \alpha_1, \ldots\}.$$

Note that $L_0^{fw} = \mathbf{D} = \mathbf{K} + \Diamond\top$ and $L_1^{fw} = \mathbf{K} + \Diamond(p \to \Box p) = \mathbf{K} + \Box p \to \Diamond\Box p$. Then $\mathcal{C}_\Diamond(\varphi)$ is the class $\mathcal{C}_\omega^{fw}$ of all frames $\mathcal{F} = (W, R)$ satisfying *possible finality condition*[1]

- for all worlds x in W, there exists a world y in $R(x)$ such that $R(y) \subseteq \{y\}$,

[1] In the transitive case it is also called "McKinsey property" [14]

i.e. every world can see a final world.

Let us also consider for every $n \geq 0$, the class $\mathcal{C}_n^{fw}$ of all frames $\mathcal{F} = (W, R)$ such that

- for all worlds x in W and for all covers $\{W_1, \ldots, W_n\}$ of $R(x)$, there exists a world y in $R(x)$ such that for all i in $\{1, \ldots, n\}$, if $y \in W_i$ then $R(y) \subseteq W_i$.

PROPOSITION 9. $\mathcal{C}_0^{fw} \supseteq \mathcal{C}_1^{fw} \ldots \supseteq \mathcal{C}_\omega^{fw}$.

Proof. Trivial. ■

4.2 Weakly condensed frames

Remark that $\mathcal{C}_0^{fw}$ is nothing but the class of all serial frames and $\mathcal{C}_1^{fw}$ is the class of all *weakly condensed* frames $\mathcal{F} = (W, R)$, i.e. such that:

- for all worlds $x \in W$, there exists $y \in R(x)$ such that $R(y) \subseteq R(x)$.

For a frame $\mathcal{F} = (W, R)$, consider the relation $R^{\rightarrow} \subseteq R$:

$$xR^{\rightarrow}y := \{y\} \cup R(y) \subseteq R(x)$$

Thus $\mathcal{F}$ is weakly condensed iff $(W, R^{\rightarrow})$ is serial. One can easily see that $R^{\rightarrow}$ is transitive: if $xR^{\rightarrow}yR^{\rightarrow}z$, then $R(z) \cup \{z\} \subseteq R(y) \subseteq R(x)$.

By a straightforward argument, $\mathcal{F}$ is weakly condensed iff $\mathcal{F} \vDash \alpha_1$. Since the Sahlqvist formula $\Box p \to \Diamond\Box p$ is an equivalent form of α_1, we obtain

LEMMA 10. *If $\mathcal{F}$ is the canonical frame for a logic $L \supseteq L_1^{fw}$, then $\mathcal{F}$ is weakly condensed.*

4.3 Completeness

THEOREM 11. *The canonical frame for L_2^{fw} is in $\mathcal{C}_\omega^{fw}$.*

Proof. Let $\mathcal{F} = (W, R)$ be the canonical frame for $L = L_2^{fw}$.

For any $x \in W$, put $x^+ := x^\Box \cup \{\Box\varphi \mid \Box\varphi \in x\}$.

CLAIM 1 For any worlds x, y in the canonical model

$$R(x) \supseteq R(y) \text{ iff } y \supseteq \{\Box\varphi \mid \Box\varphi \in x\}.$$

In fact, if $y \supseteq \{\Box\varphi \mid \Box\varphi \in x\}$ and yRz, then by the definition of R, $\Box\varphi \in x$ implies $\varphi \in z$, i.e. xRz. The other way round, if $y \not\supseteq \{\Box\varphi \mid \Box\varphi \in x\}$, then for some $\Box\varphi \in x$ we have $\Box\varphi \notin y$, and so there exists $z \in R(y)$ such that $\varphi \notin z$. But $\Box\varphi \in x$, hence $z \notin R(x)$. Therefore $R(y) \not\subseteq R(x)$. Q.e.d.

CLAIM 2 $xR^{\rightarrow}y$ iff $y \supseteq x^+$.

In fact, by definition, xRy iff $y \supseteq x^\Box$, and $R(x) \supseteq R(y)$ iff $y \supseteq \{\Box\varphi \mid \Box\varphi \in x\}$, by Claim 1. Q.e.d.

Due to Lemma 10, $\mathcal{F}$ is weakly condensed, and thus by Claim 2, x^+ is an L-consistent set for any $x \in W$.

For the proof of our theorem, consider an arbitrary world x in W. Assuming that all formulas are arranged in some fixed order φ_1, φ_2, …, we define the sequence y_0, y_1, … of L-consistent sets of formulas by induction:

- $y_0 = x^+$
- $$y_{n+1} = \begin{cases} y_n \cup \{\Box\varphi_n\} \text{ if this set is } L-\text{consistent}, \\ y_n \text{ otherwise} \end{cases}$$

Let $y' = y_0 \cup y_1 \cup \ldots$. Note that y' is L-consistent, and for all formulas φ, if $y' \cup \{\Box\varphi\}$ is L-consistent, then $\Box\varphi \in y'$. By the Lindenbaum Lemma, there exists a maximal L-consistent set y'' such that $y' \subseteq y''$. Since $y'' \supseteq x^+$, then by Claim 2, $xR^{\rightarrow}y''$.

Claim 3 If $y''R^{\rightarrow}v$ then $R(v) = R(y'')$.

Given $v \in R(y'')$ and $R(v) \subseteq R(y'')$, let us show that $R(y'') \subseteq R(v)$. So we assume $t \in R(y'')$ and show that vRt. For this we further assume $\Box\varphi \in v$ and show that $\varphi \in t$. Since v is a world in $R(x)$ such that $R(v) \subseteq R(x)$, we have $y' \subseteq v$. In fact, note that $x^{\Box} \subseteq v$ since xRv; all other formulas in y' are of the form $\Box\psi$, and $\Box\psi \in y' \subseteq y''$ implies $y'' \vDash \Box\psi$, and thus $v \vDash \Box\psi$ (since $R(v) \subseteq R(y'')$), i.e. $\Box\psi \in v$.

Hence $y' \cup \{\Box\varphi\}$ is an L-consistent set of formulas, and thus $\Box\varphi \in y'$ by our construction. Therefore $\Box\varphi \in y''$. Since $y''Rt$, we eventually obtain $\varphi \in t$. Q.e.d.

Claim 4 $card(R(y'')) \leq 1$.

In fact, let z, t be worlds in $R(y'')$ such that $z \neq t$. Then there exists a formula $\varphi \in z$ such that $\varphi \notin t$. Thus

$$u = (y'')^+ \cup \{\Diamond\varphi \rightarrow \Box\varphi\}$$

is an L-consistent set of formulas. For otherwise there exists formulas $\varphi_1, \ldots, \varphi_{m+n}$ such that $\Box\varphi_1, \ldots, \Box\varphi_{m+n} \in y''$ and

$$(1) \quad \neg(\varphi_1 \wedge \ldots \wedge \varphi_m \wedge \Box\varphi_{m+1} \wedge \ldots \wedge \Box\varphi_{m+n} \wedge (\Diamond\varphi \rightarrow \Box\varphi)) \in L.$$

Let

$$\varphi' = \varphi_1 \wedge \ldots \wedge \varphi_m, \; \varphi'' = \varphi_{m+1} \wedge \ldots \wedge \varphi_{m+n}.$$

So $\Box\varphi' \wedge \Box\varphi''$ is in y''.

On the other hand, by (1) and classical logic

$$\neg(\varphi' \wedge \Box\varphi'' \wedge (\Box\neg\varphi \vee \Box\varphi)) \in L,$$

which is equivalent to

$$\varphi' \rightarrow \neg(\Box(\varphi'' \wedge \varphi) \vee \Box(\varphi'' \wedge \neg\varphi)) \in L.$$

Hence

$$\Box\varphi' \rightarrow \Box\neg(\Box(\varphi'' \wedge \varphi) \vee \Box(\varphi'' \wedge \neg\varphi)) \in L,$$

which is equivalent to

$$(2) \quad \Box\varphi' \rightarrow \Box(\neg\Box(\varphi'' \wedge \varphi) \wedge \neg\Box(\varphi'' \wedge \neg\varphi)) \in L.$$

But

$$(3) \quad \Diamond((\varphi'' \wedge \varphi \to \Box(\varphi'' \wedge \varphi)) \wedge (\varphi'' \wedge \neg\varphi \to \Box(\varphi'' \wedge \neg\varphi))) \in L$$

as a substitution instance of α_2. Since $\Box\varphi' \in y''$, from (2) and (3) we obtain for some $w \in R(y'')$:

$$\neg(\varphi'' \wedge \varphi) \wedge \neg(\varphi'' \wedge \neg\varphi) \in w.$$

Then $\neg\varphi'' \in w$, which contradicts $\Box\varphi'' \in y''$.

Therefore u is L-consistent. By the Lindenbaum Lemma, there exists a maximal L-consistent set $u' \supseteq u$. By the above construction and Claim 2, $y''R^{\to}u'$, and also $\Diamond\varphi \to \Box\varphi \in u'$. So by Claim 3, $R(u') = R(y'')$. Therefore z, t are worlds in $R(u')$. Hence $\Diamond\varphi \in u'$, and consequently $\Box\varphi$ is in u'. Thus φ is in t: a contradiction. Q.e.d.

Since $\mathcal{F}$ is weakly condensed, for some z we have $y''R^{\to}z$, i.e., $y''Rz$ and $R(z) \subseteq R(y'')$. By Claim 4, $R(y'') = \{z\}$, thus $R(z) \subseteq \{z\}$. $xR^{\to}y''$ and $y''R^{\to}z$ implies xRz, since $R^{\to}$ is transitive and $R^{\to} \subseteq R$. Thus $\mathcal{F} \in \mathcal{C}_\omega^{fw}$. ∎

COROLLARY 12. *L_2^{fw} is strongly complete with respect to $\mathcal{C}_\omega^{fw}$.*

COROLLARY 13. *The* McKinsey formula $\Box\Diamond p \to \Diamond\Box p$ *is in* L_2^{fw}.

Proof.[2] In fact, $\mathcal{C}_\omega^{fw} \models \Box\Diamond p \to \Diamond\Box p$. ∎

PROPOSITION 14. $L_2^{fw} = L_3^{fw} \ldots = L_\omega^{fw}$.

Proof. Follows from Theorems 3 and 11.

However let us give a syntactic proof of this fact proposed by Max Cresswell in a private communication.

It is sufficient to show that $L_n^{fw} \vdash \alpha_{n+1}$ for $n \geq 2$. Let

$$\pi_1 := (p_1 \to \Box p_1), \quad \pi_2 := (p_2 \to \Box p_2) \wedge \cdots \wedge (p_n \to \Box p_n),$$
$$\pi_3 := (p_{n+1} \to \Box p_{n+1}).$$

Now we argue in L_n^{fw}:

(1)	$\Diamond(\pi_1 \wedge (\neg q \to \Box\neg q))$	$\alpha_2, Subst$
(2)	$\Diamond(\pi_1 \wedge (\Diamond q \to q))$	(1), equivalent replacement
(3)	$\Diamond(\pi_1 \wedge (\Diamond(\pi_2 \wedge \pi_3) \to (\pi_2 \wedge \pi_3)))$	(2), *Subst*
(4)	$\Diamond(\pi_2 \wedge \pi_3)$	α_n, *Subst*
(5)	$\Box\Diamond(\pi_2 \wedge \pi_3)$	(4), *Gen*
(6)	$\Diamond(p \wedge (q \to r)) \to (\Box q \to \Diamond(p \wedge r))$	derivable in **K**
(7)	$\Diamond(\pi_1 \wedge \pi_2 \wedge \pi_3)$	(6), *Subst*, (3), *MP*, (5), *MP*

Note that together with Theorem 3, this argument provides an alternative proof of Theorem 11. ∎

[2] Note that there also exists a simple syntactic proof of this corollary.

4.4 Elementarity

PROPOSITION 15. *Let $n \geq 2$. Then for any frame $\mathcal{F}$, $\mathcal{F} \in \mathcal{C}_n^{fw}$ iff $\mathcal{F} \models L_n^{fw}$.*

Proof. Let $\mathcal{F} = (W, R)$ be a frame such that $\mathcal{F} \models L_n^{fw}$ and let us show that $\mathcal{F} \in \mathcal{C}_n^{fw}$. Take an arbitrary world x in W and a cover $\{W_1, \ldots, W_n\}$ of $R(x)$. Let $\mathcal{M} = (\mathcal{F}, V)$ be a model based on $\mathcal{F}$ such that $V(p_1) = W_1$, …, $V(p_n) = W_n$. Since $x \models \alpha_n$, there exists a world y in $R(x)$ such that for all i in $\{1, \ldots, n\}$, $\mathcal{M}, y \models p_i \to \Box p_i$. Then for any i in $\{1, \ldots, n\}$, if $y \in W_i$, then $R(y) \subseteq W_i$.

Now suppose $\mathcal{F} = (W, R)$ is a frame such that $\mathcal{F} \not\models L_n^{fw}$. Then there exists a model $\mathcal{M} = (\mathcal{F}, V)$ and a world $x \in W$ such that for any $y \in R(x)$, there exists i in $\{1, \ldots, n\}$ such that $\mathcal{M}, y \not\models p_i \to \Box p_i$. Let $W_i = V(p_i)$. Then $\{W_1, \ldots, W_n\}$ is a cover of $R(x)$ and for any $y \in R(x)$ there exists i in $\{1, \ldots, n\}$ such that $y \in W_i$, but $R(y) \not\subseteq W_i$. Hence $\mathcal{F} \notin \mathcal{C}_n^{fw}$. ■

COROLLARY 16. *$\mathcal{C}_n^{fw} = \mathcal{C}_2^{fw}$ for any $n \geq 2$.*

Proof. By Propositions 14 and 15. ■

By Theorems 3, 7, and Proposition 14, L_2^{fw} is quasi-elementary and finitely elementary. The following theorem states the elementarity of L_2^{fw}.

THEOREM 17. *For any frame $\mathcal{F}$, $\mathcal{F} \in \mathcal{C}_\omega^{fw}$ iff $\mathcal{F} \models L_2^{fw}$; thus α_2 modally defines $\mathcal{C}_\omega^{fw}$.*

Proof. By Lemma 2, it is sufficient to show that $\mathbf{Fr}(L_2^{fw}) \subseteq \mathcal{C}_\omega^{fw}$.

So let $\mathcal{F} = (W, R) \models L_2^{fw}$. Then by Proposition 15 and Corollary 16, $\mathcal{F} \in \mathcal{C}_2^{fw} = \mathcal{C}_3^{fw}$. Hence we obtain

CLAIM 1 $(W, R^{\to})$ is in $\mathcal{C}_2^{fw}$.

In fact, let $x \in W$ and let $\{W_1, W_2\}$ be a cover of $R^{\to}(x)$. Also let

$$V_1 := W_1 \cap R^{\to}(x),\ V_2 := W_2 \cap R^{\to}(x),\ V_3 = R(x) \setminus (V_1 \cup V_2).$$

So $\{V_1, V_2, V_3\}$ is a cover of $R(x)$ and there exists a world y in $R(x)$ such that for all i in $\{1, 2, 3\}$, if $y \in V_i$, then $R(y) \subseteq V_i \subseteq R(x)$. Therefore $y \in R^{\to}(x)$. Next, for $i = 1, 2$ we have: if $y \in W_i$, then $y \in V_i$, so $R^{\to}(y) \subseteq R(y) \subseteq V_i \subseteq W_i$. Hence $(W, R^{\to})$ is in $\mathcal{C}_2^{fw}$. Q.e.d.

CLAIM 2 $(W, R^{\to})$ is in $\mathcal{C}_\omega^{fw}$.

In fact, $\Box\Diamond p \to \Diamond\Box p$ is in L_2^{fw} (Corollary 13); then by Proposition 15 and Claim 1, $(W, R^{\to}) \models \Box\Diamond p \to \Diamond\Box p$. Since $R^{\to}$ is transitive, $(W, R^{\to})$ is in $\mathcal{C}_\omega^{fw}$. Q.e.d.

By Claim 2, for any world x there exists a world y in $R^{\to}(x)$ such that $R^{\to}(y) \subseteq \{y\}$. Let $W_1 = \{y\}$ and $W_2 = R(y) \setminus \{y\}$. Then $\{W_1, W_2\}$ is a cover of $R(y)$, and so there exists a world $z \in R(y)$ such that for all i in $\{1, 2\}$, if $z \in W_i$, then $R(z) \subseteq W_i$.

Suppose $z \in W_2$. Then $R(z) \cup \{z\} \subseteq R(y)$. So $yR^{\rightarrow}z$ and $z \neq y$. Consequently $R^{\rightarrow}(y) \not\subseteq \{y\}$: a contradiction.

Hence $z \in W_1$, i.e. $z = y$, and so $R(z) \subseteq \{y\}$. Since $y \in R^{\rightarrow}(x) \subseteq R(x)$, y is a final world in $R(x)$. ∎

4.5 Fmp and decidability

THEOREM 18. *L_2^{fw} has the fmp and therefore is decidable.*

Proof. Given an L_2^{fw}-consistent formula φ, let us show that φ is satisfiable in some finite frame from $\mathcal{C}_\omega^{fw}$.

Let $\mathcal{M} = (W, R, V)$ be the canonical model for L_2^{fw}. Then $\mathcal{M}, x_0 \models \varphi$ for some $x_0 \in W$ and $(W, R) \in \mathcal{C}_\omega^{fw}$ by Theorem 11. Let W^f be the set of all final worlds in $\mathcal{M}$. Take a new proposition letter $q \notin cl(\varphi)$, and let U be a valuation such that for all $p \in cl(\varphi)$, $U(p) = V(p)$, and $U(q) = W^f$. For $\mathcal{N} := (W, R, U)$ we obviously have $\mathcal{N}, x_0 \models \phi$.

Let $\mathcal{N}' = (W', R', U')$ be the minimal filtration of $\mathcal{N}$ through $\Psi = cl(\varphi) \cup \{q\}$. Recall that W' is the quotient set under the equivalence relation

$$x \sim y \text{ iff the same formulas from } \Psi \text{ are true at } \mathcal{N}, x \text{ and } \mathcal{N}, y,$$

and for $\overline{y}, \overline{z} \in W'$

$$\overline{y} R' \overline{z} \ \text{ iff } \ (y_1 R z_1 \ \text{ for some } y_1 \in \overline{y},\ z_1 \in \overline{z}),$$

where $\overline{x}$ denotes the class of x modulo $\sim$. So $\mathcal{N}'$ is finite, and $\mathcal{N}', \overline{x_0} \models \varphi$, by the Filtration Lemma.

Let us show that $(W', R') \in \mathcal{C}_\omega^{fw}$. Consider $\overline{x} \in W'$. Since $(W, R) \in \mathcal{C}_\omega^{fw}$, there exists $y \in W^f \cap R(x)$. Then $\overline{x} R' \overline{y}$. Suppose $\overline{y} R' \overline{z}$, so $y_1 R z_1$ for some $y_1 \in \overline{y},\ z_1 \in \overline{z}$. Since $y \in W^f$ we have $\mathcal{N}, y \models q$, so $\mathcal{N}, y_1 \models q$ and $y_1 \in W^f$. Hence $y_1 R z_1$ implies $z_1 = y_1$, thus $\overline{z} = \overline{y}$. It follows that $R'(\overline{y}) = \{\overline{y}\}$, thus $(W', R') \in \mathcal{C}_\omega^{fw}$. ∎

5 Every world can see a deterministic world

In our second example, we put $\varphi := \Diamond p_1 \to \Box p_1$. Let

$$\begin{aligned} \beta_n &:= \varphi_\Diamond^n = \Diamond((\Diamond p_1 \to \Box p_1) \wedge \ldots \wedge (\Diamond p_n \to \Box p_n)); \\ L_\omega^{dw} &:= L_\Diamond(\varphi) = \mathbf{K} + \{\beta_0,\ \beta_1, \ldots\} \ (= KM^\infty); \\ L_n^{dw} &:= L_\Diamond^n(\varphi) = \mathbf{K} + \beta_n, \end{aligned}$$

for $n \geq 0$.

Note that L_0^{dw} is nothing but $\mathbf{K} + \Diamond\top$, and $L_1^{dw} = \mathbf{K} + \Box\Diamond p \to \Diamond\Box p$.

The class $\mathcal{C}_\omega^{dw} := \mathcal{C}_\Diamond(\varphi)$ is the class of all frames $\mathcal{F} = (W, R)$ with the following property:

- for all worlds x in W, there exists a world y in $R(x)$ such that $card(R(y)) = 1$, i.e., every world in W can see a deterministic world in $\mathcal{F}$.

Let us also consider for $n \geq 1$, the class $\mathcal{C}_n^{dw}$ of all frames $\mathcal{F} = (W, R)$ such that

- for all worlds x in W and for all sets $W_1, \ldots, W_n \subseteq W$, there exists a world y in $R(x)$ such that for all i in $\{1, \ldots, n\}$, if $R(y) \cap W_i \neq \emptyset$ then $R(y) \subseteq W_i$.

Let $\mathcal{C}_0^{dw}$ be the class of all serial frames. One can easily see that

PROPOSITION 19. $\mathcal{C}_0^{dw} \supseteq \mathcal{C}_1^{dw} \supseteq \mathcal{C}_2^{dw} \ldots \supseteq \mathcal{C}_\omega^{dw}$.

PROPOSITION 20. *Let $n \geq 1$. Then $\mathcal{C}_n^{dw} = \mathbf{Fr}(L_n^{dw})$.*

Proof. Given a frame $\mathcal{F} = (W, R) \in \mathcal{C}_n^{dw}$, let us show that for any model $\mathcal{M} = (\mathcal{F}, V)$, for any $x \in W$ we have $\mathcal{M}, x \vDash \beta_n$. Put $W_i := V(p_i)$. Then for some $y \in R(x)$ we have: for all i in $\{1, \ldots, n\}$, $R(y) \cap W_i = \varnothing$ or $R(y) \subseteq W_i$. If $\mathcal{M}, y \vDash \Diamond p_i$ then $R(y) \cap W_i \neq \varnothing$, so $R(y) \subseteq W_i$, thus $\mathcal{M}, y \vDash \Box p_i$. Hence $\mathcal{M}, x \vDash \beta_n$.

For the converse, suppose $\mathcal{F} \vDash L_n^{dw}$ and show that $\mathcal{F} \in \mathcal{C}_n^{dw}$. Let $x \in W$, $W_1, \ldots, W_n \subseteq W$. Consider a model $\mathcal{M} = (\mathcal{F}, V)$ such that $V(p_i) = W_i$, $1 \leq i \leq n$. Since $\mathcal{M}, x \vDash \beta_n$, for some $y \in R(x)$ for all $i \in \{1, \ldots n\}$ we have: $\mathcal{M}, y \vDash \Diamond p_i \to \Box p_i$, so $R(y) \cap W_i \neq \emptyset$ implies $R(y) \subseteq W_i$. Thus $\mathcal{F} \in \mathcal{C}_n^{dw}$. ■

From [7] it follows that L_ω^{dw} is not finitely-axiomatizable (this proof is based on constructions from [5], [6]).

It is well known that L_1^{dw} is not Δ-elementary; this was proved independently in [1] and [5]; a proof can also be found in [3]. Next, [7] proves that L_ω^{dw} is not Δ-elementary (and thus certainly $\mathcal{C}_\omega^{dw} \neq \mathbf{Fr}(L_\omega^{dw})$). Let us now prove the following generalization of this fact:

THEOREM 21. *If L is a modal logic, $L_1^{dw} \subseteq L \subseteq L_\omega^{dw}$, then L is not Δ-elementary.*

Proof. The main idea is the same as in [1]. First, for any Kripke model $\mathcal{M}$ and $n \geq 1$ we define the relation

$$z_1 \equiv_n^{\mathcal{M}} z_2 \;:=\; \mathcal{M}, z_1 \vDash p_i \Leftrightarrow \mathcal{M}, z_2 \vDash p_i \text{ for all } i \in \{1 \ldots n\}.$$

Clearly, $\equiv_n^{\mathcal{M}}$ is an equivalence relation on $\mathcal{M}$, and the quotient set $W/\equiv_n^{\mathcal{M}}$ is finite. By a straightforward argument,

$$\mathcal{M}, y \vDash (\Diamond p_1 \to \Box p_1) \wedge \ldots \wedge (\Diamond p_n \to \Box p_n) \text{ iff } z_1 \equiv_n^{\mathcal{M}} z_2 \text{ for all } z_1, z_2 \in R(y)$$

Now let us define a certain frame $\mathcal{F} = (W, R)$. For a countable set $Z = \{z_i \mid i \in \mathbb{N}\}$, take the uncountable set

$$Y := \{y_U \mid U \subseteq Z,\ U \text{ is infinite}\},$$

and put $W := Z \cup Y \cup \{x\}$, where $x \notin Z \cup Y$. R is defined as follows (Fig. 1):

$$R(x) := Y,\ R(y_U) := U,\ R(z_i) := \{z_i\}.$$

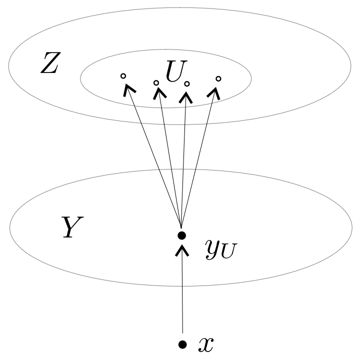

Figure 1.

CLAIM 1 $\mathcal{F} \vDash L^{dw}_{\omega}$.

It is sufficient to show that for all $n \geq 0$ and for all models $\mathcal{M}$ based on $\mathcal{F}$, $\mathcal{M}, x \vDash \beta_n$. Since $W/\equiv^{\mathcal{M}}_n$ is finite, there exists an $\equiv^{\mathcal{M}}_n$-equivalence class U_0 such that $U := U_0 \cap Z$ is infinite. Then

$$\mathcal{M}, y_U \vDash (\Diamond p_1 \to \Box p_1) \wedge \ldots \wedge (\Diamond p_n \to \Box p_n)$$

and therefore $\mathcal{M}, x \vDash \beta_n$. Q.e.d.

Now suppose the class $\mathbf{Fr}(L)$ is Δ-elementary, i.e., definable by a first-order theory T in the language $\{R, =\}$. Since $\mathcal{F} \vDash L^{dw}_{\omega}$, we have $\mathcal{F} \vDash L$, thus $\mathcal{F} \vDash T$. By the Löwenheim-Skolem theorem, there exists a countable subframe $\mathcal{F}' = (W', R')$ of $\mathcal{F}$ such that $W' \supseteq Z \cup \{x\}$ and $\mathcal{F}' \vDash T$. Then $\mathcal{F}' \vDash L$, and thus $\mathcal{F}' \vDash \beta_1$

However by [6, Theorem 1], $\neg\beta_1$ is satisfiable at a point x in a frame (W, R) if for all y in $R(x)$, $R(y)$ is infinite and $card(R(y)) \geq card(R(x))$. Thus $\mathcal{F}', x \nvDash \beta_1$.

This contradiction proves the theorem. ■

COROLLARY 22. *$\mathcal{C}^{dw}_{\omega}$ is not modally definable.*

Proof. In fact, if an elementary class of frames $\mathcal{C}$ is modally definable, then its modal logic $\mathbf{L}(\mathcal{C})$ is elementary — just because $\mathbf{Fr}(\mathbf{L}(\mathcal{C}))$ is the smallest modally definable class containing $\mathcal{C}$. ■

For $\mathbf{v} = \{v_1, \ldots, v_n\} \in \{0,1\}^n$, put $\mathbf{p}^{\mathbf{v}} := \bigwedge\limits_{1 \leq i \leq n} p_i^{v_i}$, where $p^1 := p$, $p^0 := \neg p$. It is not difficult to check that for all $n > 0$, β_n is equivalent to $\bigvee\limits_{\mathbf{v} \in \{0,1\}^n} \Diamond\Box\mathbf{p}^{\mathbf{v}}$. Thus the logics L^{dw}_1, $L^{dw}_2, \ldots, L^{dw}_{\omega}$ are uniform. Since all serial uniform logics have the fmp (see e.g. [4]), we have the following

THEOREM 23. *The logics L^{dw}_1, $L^{dw}_2, \ldots, L^{dw}_{\omega}$ have the fmp.*

COROLLARY 24. *The logics L^{dw}_1, $L^{dw}_2, \ldots, L^{dw}_{\omega}$ are decidable.*

Proof. For finite n L_n^{dw} is finitely axiomatizable. The decidability of L_ω^{dw} follows by Theorem 23 and Proposition 8. ∎

6 Complexity

It is well-known that all logics between **K** and **S4** are PSPACE-hard [12]. This result can be easily extended to all logics between **K** and $\mathbf{S4} + \Box\Diamond p \to \Diamond\Box p$ (see e.g. [15]). Thus L_2^{fw}, L_ω^{dw} are PSPACE-hard. PSPACE-decidability of the logics L_2^{fw}, L_ω^{dw} was recently obtained by the first author, the proof will be published in the sequel.

7 Conclusion

Stepping aside from a familiar field leads us to various nontrivial questions. We hope to address some of them in the second part of this paper. Let us only mention several topics for further study.

1. We see that $L_\Diamond(\varphi)$ (for a Sahlqvist formula φ) is sometimes finitely axiomatizable. Does there exist a reasonable criterion (or at least a sufficient condition) for that?

2. Basing on his investigation in first-order modal logic, Sergei Astretsov (Moscow State University) proposed the following conjecture: $L_\Diamond(\varphi)$ is finitely axiomatizable iff it is elementary. This conjecture is consistent with the three examples from the present paper.

3. Does the fmp always transfer from $\mathbf{K} + \varphi$ to $L_\Diamond(\varphi)$?

4. What are the properties of $L_\Diamond(\varphi)$ for well-known formulas φ, such as transitivity, non-branching, symmetry? As for transitivity, there is some progress: recently Stanislav Kikot (Moscow State University) has proved that $\mathcal{C}_\Diamond(\Box p \to \Box^2 p)$ is not modally definable. This leads to another question: does the elementarity of $L_\Diamond(\varphi)$ imply the modal definability of $\mathcal{C}_\Diamond(\varphi)$? Note that the converse is rather trivial, see the proof of Corollary 22.

5. The logics $L_\Diamond(\varphi)$ are not Δ-elementary for $\varphi = \Box p \to p,\ p \to \Box p$. What happens to other non-elementary logics $L_\Diamond(\varphi)$ from Theorem 3? Note that according to a result by Van Benthem [2], every *finitely axiomatizable* Δ-elementary modal logic is elementary.

6. Note that Theorem 21 on non-elementarity of "approximants" $L_\Diamond^n(\varphi)$ holds for KM^∞, but its analogue fails for Hughes' logic $L_\Diamond(\Box p \to p)$, since $L_\Diamond^1(\Box p \to p) = \mathbf{K} + \Box\Box p \to \Diamond p$ is axiomatized by a Sahlqvist formula. What happens in the general case – is it true that if $L_\Diamond(\varphi)$ (for a Sahlqvist φ) is non-elementary, then $L_\Diamond^n(\varphi)$ are also non-elementary for sufficiently large n?

7. Is it true that all the logics $L_\Diamond^n(\varphi)$ are complete (again for a Sahlqvist φ)? We do not know this even for simple cases, like $n = 2,\ \varphi = p \to \Box p$.

8. Is it possible to extend the result on non-canonicity of approximants for $L_\Diamond(\varphi)$ by Goldblatt–Hodkinson [7] to a larger class of logics $L_\Diamond(\varphi)$? What about noncompactness (for the McKinsey axiom this was established in [16])?

9. What happens in the transitive case? More exactly, consider logics of the form $\mathbf{K4} + L_\Diamond(\varphi)$ for locally elementary and locally d-persistent φ. Our

results show that they are quasi-elementary. Are they elementary? finitely axiomatizable? Note that e.g. $\mathbf{K4} + L_\diamond(p \to \Box p) = \mathbf{K4} + \Box\diamond p \to \diamond\Box p$ is elementary.

10. Theorem 3 can probably be extended to a larger class of logics. Of course, it survives in the polymodal case. Moreover, instead of the prefix $\diamond$ one can take a prefix $\diamond_{k_1} \dots \diamond_{k_n}$, with the corresponding condition "every world sees a φ-world via the composed relation $R_{k_1} \circ \dots \circ R_{k_n}$". And furthermore, we can consider more complicated conditions, like "every world sees a world seeing a φ-world and a ψ-world". It would be interesting to find a natural general result of this kind.

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