Simulation of Two Dimensions in Unimodal Logics

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Abstract

In this paper, we prove undecidability and the lack of finite model property for a certain class of unimodal logics. To do this, we adapt the technique from [7], where products of transitive modal logics were investigated, for the unimodal case. As a particular corollary, we present an undecidable unimodal fragment of Halpern and Shoham's Interval Temporal Logic.

Keywords: products of modal logics, undecidable modal logics, logics without the finite model property, locally one-component frames, Halpern and Shoham's Interval Temporal Logic.

1 Introduction

In the recent paper [7], it was shown that products of transitive modal logics are usually undecidable and lack the finite model property. In the present paper we adapt the technique from [7] for the unimodal case.

We consider logics of $\curlyvee$*-products*: if (W, R_1, R_2) is the product of frames F_1 and F_2, we put $\mathsf{F}_1 \curlyvee \mathsf{F}_2 = (W, R_1 \cup R_2)$. We show that for a certain class of frames this operation allows us to 'maintain' relations R_1 and R_2. Namely, we consider $\curlyvee$-products of transitive *locally one-component frames*: a frame (W, R) is locally one-component at a point w, if the set of all points R-accessible from w cannot be split into the disjoint union of two R-incomparable non-empty sets. In particular, if a transitive frame is linear or directed, then it is locally one-component.

We show that products of unimodal locally one-component frames can be simulated in $\curlyvee$-products. Also, by presenting a set of unimodal axioms, we define a class of unimodal frames, which allows us to 'encode' the modalities of the commutator $[\mathrm{K4}, \mathrm{K4}]$. For various unimodal logics (defined syntactically or semantically), it leads to undecidability and the lack of finite model property.

It is known that modal logics of products are related to modal logics of intervals (see e.g. [10]), namely – to fragments of Halpern and Shoham's Interval Temporal Logic HS [8]. This allows us to prove similar results for a unimodal fragment of HS. It is known that HS and many of its fragments are undecidable over various classes of intervals, for the latest results see [3,2]; these results were obtained for fragments with two or more modalities. Also, in the very recent paper [4], undecidability for a fragment of HS with a single modality was obtained: it was shown that the logic of the *'overlap'* relation is undecidable over discrete linear orders. We obtain another result of this kind: we show the undecidability and the lack of finite model property for the $\langle \overline{B} \vee \overline{E} \rangle$-fragment of HS interpreted over various classes of intervals (including intervals on real and rational numbers), where the modality $\langle \overline{B} \vee \overline{E} \rangle$ corresponds to the inverse of the union of Allen's relations *'begins'* and *'ends'* . As far as we know, this is the first example of an undecidable unimodal fragment of HS over dense linear orders.

The paper is organized as follows. In Section 2 we introduce some basic notions and notations. Section 3 contains some auxiliary observations on modal-to-modal translations that allow us to adapt the technique from [7] to the unimodal case, and also to find some modal axioms for $\curlyvee$-products. In Section 4 we study basic properties of $\curlyvee$-products of locally one-component frames. In Section 5 we formulate and prove results on undecidability and the lack of finite model property. In Section 6 we consider the $\langle \overline{B} \vee \overline{E} \rangle$-fragment of HS.

2 Preliminaries

We consider *propositional normal modal logics* with finitely many modalities. PV denotes the countable set of all propositional variables. The set of all *n-formulas* ML_n is constructed from PV, the classical connectives $\wedge$, $\neg$, and the unary connectives $\Diamond_1, \ldots, \Diamond_n$. Other connectives are regarded as abbreviations, in particular, $\Box_i \varphi = \neg \Diamond_i \neg \varphi$. $\Diamond$ and $\Box$ abbreviate $\Diamond_1$ and $\Box_1$, respectively.

Variables are typically denoted by $\mathsf{p}, \mathsf{q}, \mathsf{r}$, formulas – by φ, ψ, possibly subscripted. For a formula φ, $PV(\varphi)$ denotes the set of all variables of φ. For a set of formulas Γ, $PV(\Gamma) = \bigcup_{\varphi \in \Gamma} PV(\varphi)$. For formulas φ, ψ and a variable p, $[\varphi/\mathsf{p}]\psi$ denotes the result of substitution of φ for p in ψ. Also we use the abbreviations

$$\Diamond_\psi \varphi = \Diamond(\psi \wedge \varphi), \quad \Box_\psi \varphi = \Box(\psi \to \varphi), \quad \varphi^0 = \neg\varphi, \quad \varphi^1 = \varphi.$$

An *(n-)frame* is a tuple $\mathsf{F} = (W, R_1, \ldots, R_n)$, where $W \neq \varnothing$, $R_i \subseteq W \times W$; an *(n-)model* M *based on* F is a pair (F, θ) or a tuple $(W, R_1, \ldots, R_n, \theta)$, where $\theta : PV \to \mathcal{P}(W)$, $\mathcal{P}(W)$ is the powerset of W; θ is called *a valuation on* W. The *truth of a formula at a point in a model*, and also the *validity of a formula in a frame* (or *in a class of frames*) are defined in the standard way, see e.g. [1]. In symbols, $\mathsf{M}, w \vDash \varphi$ means that φ is true at w in M, $|\varphi|_\mathsf{M} = \{w \mid \mathsf{M}, w \vDash \varphi\}$. $\mathsf{F} \vDash \varphi$ means that φ is valid in F. $\mathsf{F}, w \vDash \varphi$ means that $(\mathsf{F}, \theta), w \vDash \varphi$ for any valuation θ. For a set of formulas Ψ, $\mathsf{F} \vDash \Psi$ means $\mathsf{F} \vDash \varphi$ for all $\varphi \in \Psi$.

φ is *satisfiable in a frame* F *at a point* w, if $(\mathsf{F},\theta), w \vDash \varphi$ for some valuation θ. For a class of frames $\mathcal{F}$, φ is *satisfiable in* $\mathcal{F}$ (or $\mathcal{F}$*-satisfiable*), if φ is satisfiable in F for some $\mathsf{F} \in \mathcal{F}$. For a logic L, if $\mathsf{F} \vDash \mathrm{L}$, we say that F is *an* L*-frame*; φ is L*-satisfiable*, if φ is satisfiable in an L-frame.

For a binary relation R on a set W, $R^=$ denotes the reflexive closure of R, i.e., $R^= = R \cup \{(w,w) \mid w \in W\}$. wRv means $(w,v) \in R$. For $w \in W$, $V \subseteq W$, put

$$R(w) = \{v \mid wRv\}, \quad R(V) = \bigcup_{w \in V} R(w).$$

For a frame $\mathsf{F} = (W, R_1, \ldots, R_n)$, R_F^{cone} denotes the transitive reflexive closure of $R_1 \cup \cdots \cup R_n$. A point w in F is called *a root of* F, if $W = R_\mathsf{F}^{cone}(w)$; in this case F is called *rooted*. F^w (M^w) denotes the subframe of F (submodel of M) generated by w, see e.g. [1].

For relations R,S, $R \circ S$ denotes their composition, $R^2 = R \circ R$, $R^{m+1} = R \circ R^m$.

$\mathsf{F} \times \mathsf{G}$ denotes *the product of frames* F, G; for logics $\mathrm{L}_1, \mathrm{L}_2$, $[\mathrm{L}_1, \mathrm{L}_2]$ denotes their *commutator*, see e.g. [5].

The following construction was used in [7] to prove negative results on products of transitive modal logics, and will also play an important role in this paper.

Fix variables h, v and put

$$\Diamond_h \varphi = \bigwedge_{\varepsilon=0,1} (\mathsf{h}^\varepsilon \to \Diamond_1(\neg \mathsf{h}^\varepsilon \wedge (\varphi \vee \Diamond_1 \varphi)),$$
$$\Diamond_v \varphi = \bigwedge_{\varepsilon=0,1} (\mathsf{v}^\varepsilon \to \Diamond_2(\neg \mathsf{v}^\varepsilon \wedge (\varphi \vee \Diamond_2 \varphi))$$

(recall that for a formula ψ, $\psi^0 = \neg\psi$, $\psi^1 = \psi$).

For a 2-model M, put

$$\bar{R}_{h,0}^{\mathsf{M}} = \{(u,w) \mid uR_1w \;\&\; (\mathsf{M}, u \vDash \mathsf{h} \Leftrightarrow \mathsf{M}, w \vDash \neg \mathsf{h})\},$$
$$\bar{R}_{v,0}^{\mathsf{M}} = \{(u,w) \mid uR_2w \;\&\; (\mathsf{M}, u \vDash \mathsf{v} \Leftrightarrow \mathsf{M}, w \vDash \neg \mathsf{v})\},$$
$$\bar{R}_{h}^{\mathsf{M}} = \bar{R}_{h,0}^{\mathsf{M}} \cup \left(\bar{R}_{h,0}^{\mathsf{M}}\right)^2, \quad \bar{R}_{v}^{\mathsf{M}} = \bar{R}_{v,0}^{\mathsf{M}} \cup \left(\bar{R}_{v,0}^{\mathsf{M}}\right)^2,$$

where R_1, R_2 are the accessability relations of M. Clearly,

$$\bar{R}_{h}^{\mathsf{M}} = \{(u,w) \mid \exists u' \in R_1(u)\, (w \in R_1^=(u') \& (\mathsf{M}, u \vDash \mathsf{h} \Leftrightarrow \mathsf{M}, u' \vDash \neg \mathsf{h}))\},$$
$$\bar{R}_{v}^{\mathsf{M}} = \{(u,w) \mid \exists u' \in R_2(u)\, (w \in R_2^=(u') \& (\mathsf{M}, u \vDash \mathsf{v} \Leftrightarrow \mathsf{M}, u' \vDash \neg \mathsf{v}))\}.$$

For any w in M, $\varphi \in ML_2$, we have

$$\mathsf{M}, w \vDash \Diamond_h \varphi \Leftrightarrow \exists u \in \bar{R}_{h}^{\mathsf{M}}(w)(\mathsf{M}, u \vDash \varphi), \quad \mathsf{M}, w \vDash \Diamond_v \varphi \Leftrightarrow \exists u \in \bar{R}_{v}^{\mathsf{M}}(w)(\mathsf{M}, u \vDash \varphi)$$

(see [7] for more details).

Put

$$\psi_{hv} = \bigwedge_{\varepsilon=0,1} \Box_1\Box_2 \left((\mathsf{h}^\varepsilon \vee \Diamond_2 \mathsf{h}^\varepsilon \to \Box_2 \mathsf{h}^\varepsilon) \wedge (\mathsf{v}^\varepsilon \vee \Diamond_1 \mathsf{v}^\varepsilon \to \Box_1 \mathsf{v}^\varepsilon) \right).$$

Proposition 2.1 ([7]) *If* M *is a model based on a* [K4, K4]*-frame with a root* w*, and* $\mathsf{M}, w \vDash \psi_{hv}$*, then* $(W, \bar{R}_h^{\mathsf{M}}, \bar{R}_v^{\mathsf{M}})$ *is a* [K4, K4]*-frame.*

Proof. Straightforward. See [7] for more details. □

3 Modally definable relations in pretransitive frames

In [7] various undecidable problems and infiniteness of a model are encoded by formulas of the form $\psi_{hv} \wedge \psi$, where ψ is built using propositional variables, boolean connectives and derived modal operators $\Diamond_v, \Diamond_h$. Our goal is to describe a unimodal analogue of this fragment.

In this section we consider some syntactic constructions which will be used later to transfer negative results about products to the unimodal case.

3.1 'Diamond-like' formulas

Fix a variable $\mathsf{s} \in PV$. For any formulas ψ, φ, put $\psi(\varphi) = [\varphi/\mathsf{s}]\psi$. Given an n-model $\mathsf{M} = (\mathsf{F}, \theta)$, with every formula $\psi \in ML_n$ we associate a function $\psi^{\mathsf{M}} : 2^W \to 2^W$ defined in the following way:

$$\mathsf{s}^{\mathsf{M}}(V) = V, \qquad \mathsf{p}^{\mathsf{M}}(V) = \theta(\mathsf{p}), \text{ if } \mathsf{p} \in PV \text{ and } \mathsf{p} \neq \mathsf{s},$$
$$(\neg\psi)^{\mathsf{M}}(V) = W - \psi^{\mathsf{M}}(V), \qquad (\psi_1 \wedge \psi_2)^{\mathsf{M}}(V) = \psi_1^{\mathsf{M}}(V) \cap \psi_2^{\mathsf{M}}(V),$$
$$(\Diamond_i \psi)^{\mathsf{M}}(V) = R_i^{-1}(\psi^{\mathsf{M}}(V)).$$

Clearly, for any $\varphi \in ML_n$,

$$\psi^{\mathsf{M}}(|\varphi|_{\mathsf{M}}) = |\psi(\varphi)|_{\mathsf{M}}.$$

Definition 3.1 Consider an n-model M and a relation $\widetilde{R} \subseteq W \times W$. We say that a formula $\psi \in ML_n$ *expresses* $\widetilde{R}$ *in* M, in symbols $\psi \xrightarrow{\mathsf{M}} \widetilde{R}$, if

$$\psi^{\mathsf{M}}(V) = \widetilde{R}^{-1}(V) \text{ for any } V \subseteq W. \tag{1}$$

We say that ψ *expresses* $\widetilde{R}$ *in* F, in symbols $\psi \xrightarrow{\mathsf{F}} \widetilde{R}$, if (1) holds for any M based on F.

Proposition 3.2 *For an* n*-model* $\mathsf{M} = (\mathsf{F}, \theta)$*, the following conditions are equivalent:*

(1) $\psi \xrightarrow{\mathsf{M}} \widetilde{R}$*;*

(2) if $\theta' : PV \to \mathcal{P}(W)$*,* $\theta'(p) = \theta(p)$ *for any* $p \in (PV(\psi) - \{\mathsf{s}\})$*, then for any* $w \in W$*,* $\varphi \in ML_n$*, we have*

$$(F, \theta'), w \vDash \psi(\varphi) \ \Leftrightarrow \ \exists u \in \widetilde{R}(w)((F, \theta'), u \vDash \varphi).$$

Proof. Let N denote (F, θ').

(1) ⇒ (2). If $\theta'(p) = \theta(p)$ for any $p \in (PV(\psi) - \{\mathsf{s}\})$, then φ^{M} and φ^{N} are the same functions, so $|\psi(\varphi)|_{\mathsf{N}} = \psi^{\mathsf{N}}(|\varphi|_{\mathsf{N}}) = \psi^{\mathsf{M}}(|\varphi|_{\mathsf{N}}) = \widetilde{R}^{-1}(|\varphi|_{\mathsf{N}})$.

(2) ⇒ (1). For $V \subseteq W$, put $\theta'(\mathsf{s}) = V$, $\theta'(\mathsf{p}) = \theta(\mathsf{p})$ for $\mathsf{p} \neq \mathsf{s}$. We have: $\psi^{\mathsf{M}}(V) = \psi^{\mathsf{N}}(V) = |\psi|_{\mathsf{N}} = \widetilde{R}^{-1}(|\mathsf{s}|_{\mathsf{N}}) = \widetilde{R}^{-1}(V)$. □

Example 3.3 Consider a model M based on a 2-frame F. Recall that the operators $\Diamond_h$, $\Diamond_v$ and the relations $\bar{R}^{\mathsf{M}}_h, \bar{R}^{\mathsf{M}}_v$ are associated in the following way:

$$\mathsf{M}, w \vDash \Diamond_h \varphi \Leftrightarrow \exists u \in \bar{R}^{\mathsf{M}}_h(w)(\mathsf{M}, u \vDash \varphi), \quad \mathsf{M}, w \vDash \Diamond_v \varphi \Leftrightarrow \exists u \in \bar{R}^{\mathsf{M}}_v(w)(\mathsf{M}, u \vDash \varphi).$$

Moreover, the above equivalences hold for any model based on F, if its valuation on h, v is the same as in M. In other words, for the formulas

$$\psi_h = \bigwedge_{\varepsilon=0,1} (\mathsf{h}^\varepsilon \to \Diamond_1(\neg \mathsf{h}^\varepsilon \wedge (\mathsf{s} \vee \Diamond_1 \mathsf{s}))), \ \psi_v = \bigwedge_{\varepsilon=0,1} (\mathsf{v}^\varepsilon \to \Diamond_2(\neg \mathsf{v}^\varepsilon \wedge (\mathsf{s} \vee \Diamond_2 \mathsf{s}))),$$

we have

$$\psi_h \stackrel{\mathsf{M}}{\rightharpoondown} \bar{R}^{\mathsf{M}}_h, \quad \psi_v \stackrel{\mathsf{M}}{\rightharpoondown} \bar{R}^{\mathsf{M}}_v. \tag{2}$$

This example is important for us: the proofs of our negative results are based on the fact that the relations $\bar{R}^{\mathsf{M}}_h$, $\bar{R}^{\mathsf{M}}_v$ can be expressed in the unimodal language.

To describe fragments of modal logics in different languages it is convenient to use the following construction.

Definition 3.4 Given formulas $\psi_1, \ldots \psi_k \in ML_n$, let $[\ \ ]_{(\psi_1,\ldots\psi_k)}$ denote the following translation from ML_k to ML_n:

$$[\mathsf{p}]_{(\psi_1,\ldots\psi_k)} = \mathsf{p} \text{ for } \mathsf{p} \in PV;$$
$$[\phi \wedge \psi]_{(\psi_1,\ldots\psi_k)} = [\phi]_{(\psi_1,\ldots\psi_k)} \wedge [\psi]_{(\psi_1,\ldots\psi_k)};$$
$$[\neg\phi]_{(\psi_1,\ldots\psi_k)} = \neg([\phi]_{(\psi_1,\ldots\psi_k)});$$
$$[\Diamond_i \phi]_{(\psi_1,\ldots\psi_k)} = \psi_i([\phi]_{(\psi_1,\ldots\psi_k)}).$$

This definition is explained by the following simple lemmas.

Lemma 3.5 *Consider a model* $\mathsf{M} = (W, R_1, \ldots, R_n, \theta)$ *and relations* $\widetilde{R}_1, \ldots, \widetilde{R}_k \subseteq W \times W$*. Let* $\varphi \in ML_k$*,* $\psi_1^\diamond, \ldots \psi_k^\diamond \in ML_n$*,*

$$\psi_1^\diamond \stackrel{\mathsf{M}}{\rightharpoondown} \widetilde{R}_1, \ldots, \psi_k^\diamond \stackrel{\mathsf{M}}{\rightharpoondown} \widetilde{R}_k.$$

Let θ' be a valuation such that $\theta'(\mathsf{p}) = \theta(\mathsf{p})$ for any $\mathsf{p} \in PV(\varphi, \psi_1^\diamond, \ldots \psi_k^\diamond)$. Then for any $w \in W$, we have

$$\mathsf{M}, w \vDash [\varphi]_{(\psi_1^\diamond, \ldots \psi_k^\diamond)} \quad \Leftrightarrow \quad (W, \widetilde{R}_1, \ldots, \widetilde{R}_k, \theta'), w \vDash \varphi.$$

Proof. By induction on the construction of φ. The basis is trivial.

Suppose $\varphi = \Diamond_i \chi$. Then $[\varphi]_{(\psi_1^\diamond, \ldots \psi_k^\diamond)} = \psi_i^\diamond([\chi]_{(\psi_1^\diamond, \ldots \psi_k^\diamond)})$. We have:

$$\begin{aligned} \mathsf{M}, w \vDash \psi_i^\diamond([\chi]_{(\psi_1^\diamond, \ldots \psi_k^\diamond)}) &\Leftrightarrow \mathsf{M}, v \vDash [\chi]_{(\psi_1^\diamond, \ldots \psi_k^\diamond)} \text{ for some } v \in \widetilde{R}_i(w) \text{ (by Proposition 3.2)} \\ &\Leftrightarrow (W, \widetilde{R}_1, \ldots, \widetilde{R}_k, \theta'), w \vDash \Diamond_i \chi \text{ (by the induction hypothesis).} \end{aligned}$$

Other cases are trivial. □

Lemma 3.6 *Consider models $\mathsf{M}' = (W, R_1', \ldots, R_n', \theta)$, $\mathsf{M}'' = (W, R_1'', \ldots, R_m'', \theta)$, and relations $R_1, \ldots, R_k \subseteq W \times W$. Let $\psi_1^\diamond, \ldots \psi_k^\diamond \in ML_n$, $\phi_1^\diamond, \ldots \phi_k^\diamond \in ML_m$,*

$$\psi_i^\diamond \overset{\mathsf{M}'}{\rightharpoondown} R_i, \quad \phi_i^\diamond \overset{\mathsf{M}''}{\rightharpoondown} R_i.$$

Then for any $\varphi \in ML_k$, $w \in W$, we have

$$\mathsf{M}', w \vDash [\varphi]_{(\psi_1^\diamond, \ldots \psi_k^\diamond)} \Leftrightarrow \mathsf{M}'', w \vDash [\varphi]_{(\phi_1^\diamond, \ldots \phi_k^\diamond)}.$$

Proof. Put $\mathsf{M} = (W, R_1, \ldots, R_k, \theta)$. By Lemma 3.5,

$$\mathsf{M}', w \vDash [\varphi]_{(\psi_1^\diamond, \ldots \psi_k^\diamond)} \Leftrightarrow \mathsf{M}, w \vDash \varphi, \quad \mathsf{M}'', w \vDash [\varphi]_{(\phi_1^\diamond, \ldots \phi_k^\diamond)} \Leftrightarrow \mathsf{M}, w \vDash \varphi.$$

□

3.2 Pretransitive frames and cone formulas

Definition 3.7 [6] A frame $\mathsf{F} = (W, R_1, \ldots, R_n)$ is called *pretransitive*, if there exists a formula $\psi \in ML_n$ such that $\psi \overset{\mathsf{F}}{\rightharpoondown} R_{\mathsf{F}}^{cone}$; ψ is called *a cone formula for* F.

For a formula ψ, let

$$\psi^{(0)} = \mathsf{s}, \ \psi^{(1)} = \psi, \ \psi^{(i+1)} = \psi(\psi^{(i)}), \ \psi^{\le n} = \psi^{(n)} \vee \ldots \vee \psi^1 \vee \psi^0.$$

Example 3.8 Clearly, if $\mathsf{F} = (W, R)$ is a transitive frame, then $(\Diamond \mathsf{s})^{\le 1}$ $(= \Diamond \mathsf{s} \vee \mathsf{s})$ is a cone-formula for F. If F is a product of two transitive 1-frames, then $(\Diamond_1 \mathsf{s} \vee \Diamond_2 \mathsf{s})^{\le 2}$ is a cone formula for F. These observations are a particular case of the following proposition.

Proposition 3.9 ([6]) *An n-frame F is pretransitive iff there exists l such that $(\Diamond_1 \mathsf{s} \vee \ldots \vee \Diamond_n \mathsf{s})^{\le l}$ is a cone formula for F.*

For a pretransitive n-frame F, put $\psi_{\mathsf{F}}^{cone} = (\Diamond_1 \mathsf{s} \vee \ldots \vee \Diamond_n \mathsf{s})^{\le l_0}$, $\Diamond_{\mathsf{F}}^{cone} \varphi = \psi_{\mathsf{F}}^{cone}(\varphi)$, $\Box_{\mathsf{F}}^{cone} \varphi = \neg \psi_{\mathsf{F}}^{cone}(\neg \varphi)$, where

$$l_0 = \min\{l \mid (\Diamond_1 \mathsf{s} \vee \ldots \vee \Diamond_n \mathsf{s})^{\leq l} \text{ is a cone formula for } \mathsf{F}\}.$$

The following lemma shows how modally definable properties transfer between expressible relations in pretransitive frames.

Lemma 3.10 *Let* $\mathsf{F} = (W, R_1, \ldots, R_n)$ *be a pretransitive frame with a root* w, $\chi, \psi_1^\Diamond, \ldots, \psi_k^\Diamond \in ML_n$, $\varphi \in ML_k$, $PV(\varphi) \cap PV(\chi, \psi_1^\Diamond, \ldots, \psi_k^\Diamond) = \varnothing$. *Suppose* $R_1^\theta, \ldots, R_k^\theta \subseteq W \times W$ *for any valuation* θ *on* W, *and if* $(\mathsf{F}, \theta), w \vDash \chi$, *then*

$$\psi_1^\Diamond \overset{(\mathsf{F},\theta)}{\rightharpoondown} R_1^\theta, \ldots, \psi_k^\Diamond \overset{(\mathsf{F},\theta)}{\rightharpoondown} R_k^\theta.$$

Then the following conditions are equivalent:

(1) $\mathsf{F}, w \vDash \chi \to \Box_{\mathsf{F}}^{cone}[\varphi]_{(\psi_1^\Diamond, \ldots, \psi_k^\Diamond)}$;

(2) for any θ, *if* $(\mathsf{F}, \theta), w \vDash \chi$, *then* $\left(W, R_1^\theta, \ldots, R_k^\theta\right) \vDash \varphi$.

Proof. Let $\tilde{\mathsf{F}}^\theta$ denote $(W, R_1^\theta, \ldots, R_k^\theta)$. Put $PV_0 = PV(\chi, \psi_1^\Diamond, \ldots, \psi_k^\Diamond)$.

(1) $\Rightarrow$ (2). Let $(\mathsf{F}, \theta), w \vDash \chi$. Suppose that $(\tilde{\mathsf{F}}^\theta, \theta'), u \vDash \neg\varphi$ for some θ', u. Let η coincide with θ' on $PV(\varphi)$, and with θ on all other variables. Then $(\tilde{\mathsf{F}}^\theta, \eta), u \vDash \neg\varphi$ and $(\mathsf{F}, \eta), w \vDash \chi$. Thus $(\mathsf{F}, \eta), w \vDash \Box_{\mathsf{F}}^{cone}[\varphi]_{(\psi_1^\Diamond, \ldots, \psi_k^\Diamond)}$ and $(\mathsf{F}, \eta), u \vDash [\varphi]_{(\psi_1^\Diamond, \ldots, \psi_k^\Diamond)}$. Since $PV_0 \cap \varphi = \varnothing$, then

$$R_1^\theta = R_1^\eta, \ldots, R_k^\theta = R_k^\eta, \text{ and } \psi_1^\Diamond \overset{(\mathsf{F},\eta)}{\rightharpoondown} R_1^\theta, \ldots, \psi_k^\Diamond \overset{(\mathsf{F},\eta)}{\rightharpoondown} R_k^\theta.$$

By Lemma 3.5, $(\tilde{\mathsf{F}}^\theta, \eta), u \vDash \varphi$, which is a contradiction.

(1) $\Leftarrow$ (2). Suppose $(\mathsf{F}, \theta), w \vDash \chi$, $u \in W$. φ is valid in $\tilde{\mathsf{F}}^\theta$, so $(\tilde{\mathsf{F}}^\theta, \theta), u \vDash \varphi$, and by Lemma 3.5 $(\mathsf{F}, \theta), u \vDash [\varphi]_{(\psi_1^\Diamond, \ldots, \psi_k^\Diamond)}$. Therefore $(\mathsf{F}, \theta), w \vDash \Box_{\mathsf{F}}^{cone}[\varphi]_{(\psi_1^\Diamond, \ldots, \psi_k^\Diamond)}$. $\Box$

4 $\curlyvee$-products

4.1 Definition and basic properties

Recall that the product of 1-frames (W', R') and (W'', R'') is the frame $(W' \times W'', R_h, R_v)$, where

$(u_1, w_1) R_h (u_2, w_2) \Leftrightarrow (u_1 R' u_2 \& w_1 = w_2)$,

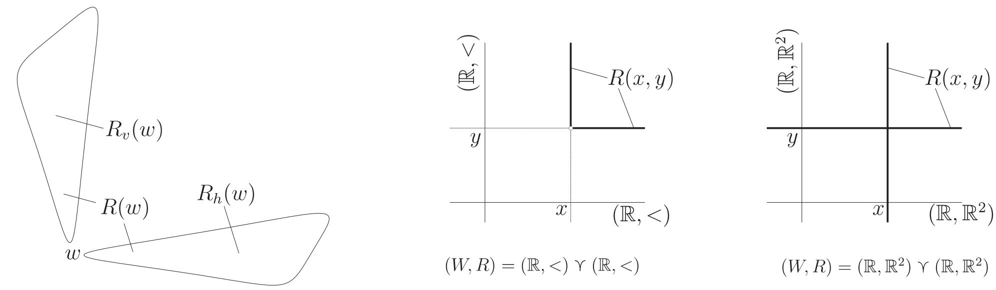
$(u_1, w_1) R_v (u_2, w_2) \Leftrightarrow (u_1 = u_2 \& w_1 R'' w_2)$.

We consider a monomodal analogue of this operation, the $\curlyvee$-*product.*[1]

Definition 4.1 The $\curlyvee$-product of frames $\mathsf{F}' = (W', R')$ and $\mathsf{F}'' = (W'', R'')$ is the frame $\mathsf{F}' \curlyvee \mathsf{F}'' = (W' \times W'', R)$, where

$$(u_1, w_1) R (u_2, w_2) \Leftrightarrow (u_1 R' u_2 \& w_1 = w_2) \text{ or } (u_1 = u_2 \& w_1 R'' w_2).$$

[1] If frames are considered as transition systems, this operation is called the *asynchronous product*, see e.g. [11].



(a) (b) (c)

Fig. 1.

Equivalently, if (W, R_h, R_v) is the product of F' and F'', then $\mathsf{F}' \curlyvee \mathsf{F}'' = (W, R_h \cup R_v)$, Fig. 1a.

For classes $\mathcal{F}_1, \mathcal{F}_2$ of n-frames, put $\mathcal{F}_1 \curlyvee \mathcal{F}_2 = \{\mathsf{F}_1 \curlyvee \mathsf{F}_2 \mid \mathsf{F}_1 \in \mathcal{F}_1, \mathsf{F}_2 \in \mathcal{F}_2\}$.

For logics $\mathrm{L}_1, \mathrm{L}_2$, put $\mathrm{L}_1 \curlyvee \mathrm{L}_2 = \mathbf{L}(\{\mathsf{F} \mid \mathsf{F} \vDash \mathrm{L}_1\} \curlyvee \{\mathsf{F} \mid \mathsf{F} \vDash \mathrm{L}_2\})$.

Example 4.2 If $(W,R) = (\mathbb{R},<) \curlyvee (\mathbb{R},<)$, then $R(x,y)$ is the union of two open rays $\{(x,t) \mid t > y\} \cup \{(t,y) \mid t > x\}$, Fig. 1b. If $(W,R) = (\mathbb{R},\mathbb{R}^2) \curlyvee (\mathbb{R},\mathbb{R}^2)$, then $R(x,y) = \{(t_1,t_2) \mid t_1 = x \text{ or } t_2 = y\}$, Fig. 1c.

These simple examples have a natural geometric interpretation. Recall the notion of the *lightlike* relation λ in Minkowski space $\mathbb{R}^n$, $n \geq 2$: $\overline{x}\lambda\overline{y} \Leftrightarrow \Sigma_{i=1}^{n-1}(y_i - x_i)^2 = (y_n - x_n)^2$, where $\overline{x} = (x_1, \ldots, x_n)$, $\overline{y} = (y_1, \ldots, y_n)$. It is easy to see that $(\mathbb{R}^2, \lambda)$ is isomorphic to the frame $(\mathbb{R},\mathbb{R}^2) \curlyvee (\mathbb{R},\mathbb{R}^2)$ (a detailed discussion of the connection between relativistic modalities and modal logics of various geometric structures can be found in [12]). Similarly, $(\mathbb{R},<) \curlyvee (\mathbb{R},<)$ is isomorphic to the frame $(\mathbb{R}^2, \lambda^\uparrow)$, where $\overline{x}\lambda^\uparrow\overline{y} \Leftrightarrow \overline{x}\lambda\overline{y} \& y_n > x_n$ (*future directed lightlike* relation).

Since $\mathrm{S5} \times \mathrm{S5}$ is decidable and has the product finite model property (see e.g. [5]), using the above observation, it is easy to show that the logic $\mathrm{L}(\mathbb{R}^2, \lambda)$ is also decidable and has the finite model property. At the same time, it follows from Theorems 5.9 and 5.14 (proved in the next sections) that the logic $\mathrm{L}(\mathbb{R}^2, \lambda^\uparrow)$ is undecidable and does not have the finite model property.

Consider some basic properties of $\curlyvee$-products.

Trivially, $(\mathsf{F}_1 \curlyvee \mathsf{F}_2, \theta), w \vDash \Diamond \mathsf{p} \Leftrightarrow (\mathsf{F}_1 \times \mathsf{F}_2, \theta), w \vDash \Diamond_1 \mathsf{p} \vee \Diamond_2 \mathsf{p}$, so $\mathbf{L}(\mathsf{F}_1) \curlyvee \mathbf{L}(\mathsf{F}_2)$ can be regarded as a fragment of $\mathbf{L}(\mathsf{F}_1 \times \mathsf{F}_2)$. In the next sections we show that these fragments can be very expressive if factors are transitive, so $\curlyvee$-products of many extensions of K4 are quite complex. But first let us consider $\curlyvee$-products of weak logics, like K, $\mathrm{T} = \mathrm{K} + \Diamond p \to p$, $\mathrm{D} = \mathrm{K} + \Diamond \top$.

Recall that w is *serial* in a frame (W,R), if $R(w) \neq \varnothing$; F is *serial*, if all its points is serial.

Due to the Definition 4.1, we have

Proposition 4.3 *For* 1*-frames* F *and* G*, we have:*

- *if one of these frames is serial (reflexive), then* $\mathsf{F} \curlyvee \mathsf{G}$ *is serial (reflexive);*
- *if* G *is an irreflexive singleton, then* $\mathsf{F} \curlyvee \mathsf{G}$ *is isomorphic to* F*;*
- *if* G *is a reflexive singleton, then* $\mathsf{F} \curlyvee \mathsf{G}$ *is isomorphic to the reflexive closure of* F*.*

These observations yield the following fact for $\curlyvee$-products of unimodal non-transitive logics.

Theorem 4.4

(i) *If an irreflexive singleton validates a logic* L*, then* $\mathrm{K} \curlyvee \mathrm{L} = \mathrm{K}$*,* $\mathrm{D} \curlyvee \mathrm{L} = \mathrm{D}$*.*

(ii) *If* $\mathrm{L}_1 \subseteq \mathrm{T}$*,* L_2 *is consistent, and* $\mathrm{L}_1 \cup \mathrm{L}_2 \supseteq \mathrm{T}$*, then* $\mathrm{L}_1 \curlyvee \mathrm{L}_2 = \mathrm{T}$*; in particular,* $\mathrm{T} \curlyvee \mathrm{L} = \mathrm{T}$ *for any consistent* L*.*

Proof. (i) Clearly, $\mathrm{K} \curlyvee \mathrm{L} \supseteq \mathrm{K}$, and $\mathrm{D} \curlyvee \mathrm{L} \supseteq \mathrm{D}$ by Proposition 4.3.

To prove the other inclusions, suppose that a formula φ is satisfiable in some (serial) frame F. Consider an irreflexive one-point frame F_0. By Proposition 4.3, $\mathsf{F} \curlyvee \mathsf{F}_0$ is isomorphic to F, so φ is $\mathrm{K}\curlyvee\mathrm{L}$-satisfiable ($\mathrm{D}\curlyvee\mathrm{L}$-satisfiable). Thus $\mathrm{K}\curlyvee\mathrm{L} \subseteq \mathrm{K}$, $\mathrm{D}\curlyvee\mathrm{L} \subseteq \mathrm{D}$.

(ii). Since $\mathrm{L}_1 \cup \mathrm{L}_2 \supseteq \mathrm{T}$, it follows that $\mathrm{L}_1 \curlyvee \mathrm{L}_2 \supseteq \mathrm{T}$ by Proposition 4.3.

To show that $\mathrm{L}_1 \curlyvee \mathrm{L}_2 \subseteq \mathrm{T}$, suppose that a formula φ is satisfiable in some reflexive frame F. Since L_2 is consistent, a one-point frame F_0 validates L_2 (Makinson's Theorem, see e.g. [1]). Since F is reflexive, by Proposition 4.3 we obtain that $\mathsf{F} \curlyvee \mathsf{F}_0$ is isomorphic to F. It follows that $\mathrm{L}_1 \curlyvee \mathrm{L}_2 \supseteq \mathrm{T}$. □

Further on, we will focus on $\curlyvee$-products of unimodal transitive frames and logics.

Consider a 1-frame (W, R). Recall that R is transitive, if $R^2 \subseteq R$. We say that (W, R) is *m-transitive*, if $R^{m+1} \subseteq R^m$. Trivially,

$$\mathsf{F} \text{ is } m\text{-transitive} \;\Leftrightarrow\; \mathsf{F} \vDash \Diamond^{m+1}\mathsf{p} \to \Diamond^m \mathsf{p}.$$

Proposition 4.5 *If* F_1 *and* F_2 *are transitive* 1*-frames, then* $\mathsf{F}_1 \curlyvee \mathsf{F}_2$ *is* 2*-transitive.*

Proof. Let $\mathsf{F}_1 \times \mathsf{F}_2 = (W, R_1, R_2)$. Due to commutativity and transitivity,

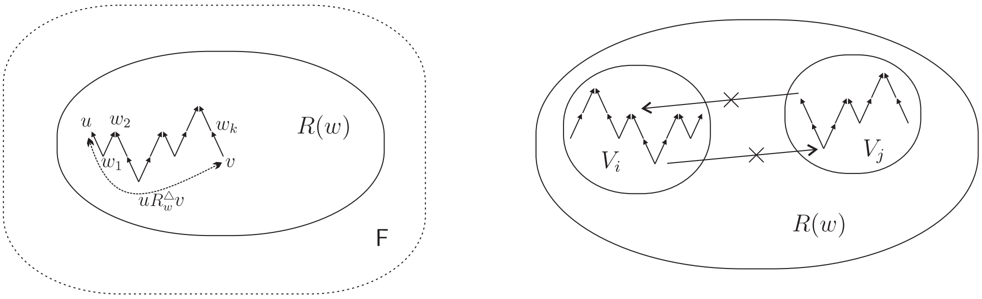
$$(R_1 \cup R_2)^3 = R_1^3 \cup (R_1^2 \circ R_2) \cup (R_1 \circ R_2^2) \cup R_2^3 \subseteq R_1^2 \cup (R_1 \circ R_2) \cup R_2^2 = (R_1 \cup R_2)^2.$$

□

4.2 Locally n-component frames

Locally n-component frames were studied in [13], and later in [9], in the context of topological modal logics. We use this notion to express the relations $\bar{R}_h^{\mathsf{M}}$, $\bar{R}_v^{\mathsf{M}}$ by unimodal formulas.

Definition 4.6 ([13]) Consider a 1-frame $\mathsf{F} = (W, R)$. For $w \in W$, let $R_w^{\triangle}$ be the following equivalence relation on $R(w)$: $uR_w^{\triangle}v$ iff there exist points $w_0, \ldots, w_{k+1} \in R(w)$ such that $u = w_0$, $w_{k+1} = v$ and for every $i = 0, \ldots, k$ we have $w_i R w_{i+1}$ or $w_{i+1} R w_i$ or $w_i = w_{i+1}$, see Fig. 2a.



(a) (b)

Fig. 2.

For serial w, let $comp_{\mathsf{F}}(w)$ denote the quotient set of $R(w)$ by $R^{\triangle}_w$; if $R(w) = \varnothing$, put $comp_{\mathsf{F}}(w) = \varnothing$. $\#_{\mathsf{F}}(w)$ denotes the cardinality of $comp_{\mathsf{F}}(w)$. F is *locally n-component*, if $\#_{\mathsf{F}}(w) \leq n$ for all $w \in W$.

If $\#_{\mathsf{F}}(w)$ is finite, then $\#_{\mathsf{F}}(w)$ is the maximal k such that for some non-empty $V_1, \ldots, V_k$ we have $R(w) = V_1 \cup \cdots \cup V_k$ and

$$\bigwedge_{1 \leq i \neq j \leq k} (V_i \cap V_j = R(V_i) \cap V_j = R(V_j) \cap V_i = \varnothing) \text{ (Fig. 2b).}$$

In this case $comp_{\mathsf{F}}(w) = \{V_1, \ldots, V_k\}$.

The above described properties are modally definable. For $n \geq 1$, put:

$$\text{COMP}(\mathsf{p}_1, \ldots, \mathsf{p}_n) = \bigwedge_{1 \leq i \leq n} \Diamond \mathsf{p}_i \wedge \Box \bigvee_{1 \leq i \leq n} \mathsf{p}_i \wedge \Box \bigwedge_{1 \leq i \neq j \leq n} (\mathsf{p}_i \to \neg(\mathsf{p}_j \vee \Diamond \mathsf{p}_j));$$

$$\text{AxComp}_n = \neg\text{COMP}(\mathsf{p}_1, \ldots, \mathsf{p}_{n+1}).$$

Proposition 4.7 ([13]) *Consider a frame* $\mathsf{F} = (W, R)$. *For any* $w \in W$, $n > 0$, *we have:*

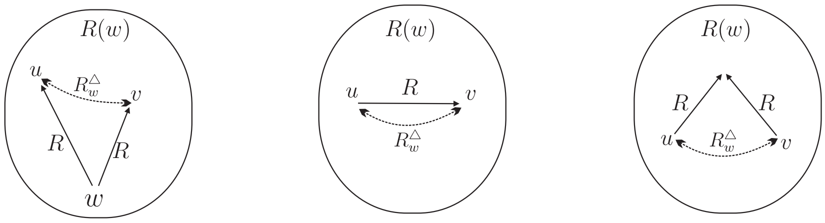
(i) $\#_{\mathsf{F}}(w) \leq n$ *iff* $\mathsf{F}, w \vDash \text{AxComp}_n$; *in particular,* F *is locally n-component iff* $\mathsf{F} \vDash \text{AxComp}_n$;

(ii) $\#_{\mathsf{F}}(w) = n$ *iff* $\mathsf{F}, w \vDash \text{AxComp}_n$ *and* $\text{COMP}_n(\mathsf{p}_1, \ldots, \mathsf{p}_n)$ *is satisfiable at* w *in* F.

Locally 1-component frames are especially important for us. Note that frames with properties like reflexivity, linearity or Church–Rosser property (the latter in the transitive case only) are locally 1-component (see Fig. 3).

Now we formulate a number of straightforward propositions that will be used in the next sections.

Proposition 4.8 *Consider* 1*-frames* F, G *and points* u *in* F *and* v *in* G. *If* u *is reflexive, then* $\#_{\mathsf{F}}(u) = \#_{\mathsf{F} \curlyvee \mathsf{G}}(u, v) = 1$; *if* u *and* v *are irreflexive, then* $\#_{\mathsf{F} \curlyvee \mathsf{G}}(u, v) = \#_{\mathsf{F}}(u) + \#_{\mathsf{G}}(v)$.



(a) (b) (c)

Fig. 3.

Proof. If u is reflexive, then (u, v) is reflexive in $\mathsf{F} \curlyvee \mathsf{G}$.

If u and v are irreflexive, then

$$comp_{\mathsf{F}\curlyvee\mathsf{G}}(u, v) = \{\{u\} \times V \mid V \in comp_{\mathsf{G}}(v)\} \cup \{\{v\} \times U \mid U \in comp_{\mathsf{F}}(u)\}.$$

□

Proposition 4.9 *If* L_1, L_2 *are unimodal logics,* $\textsc{AxComp}_n \in \mathrm{L}_1$, $\textsc{AxComp}_m \in \mathrm{L}_2$, *then* $\textsc{AxComp}_{m+n} \in \mathrm{L}_1 \curlyvee \mathrm{L}_2$.

Proof. Follows from Proposition 4.8. □

Put

$$\textsc{AxCov} \;=\; comp(\mathsf{p}, \mathsf{q}) \to (\Diamond\Diamond \mathsf{t} \wedge \neg\Diamond \mathsf{t} \to \Diamond(\mathsf{p} \wedge \Diamond \mathsf{t})).$$

By a straightforward argument, we have

Proposition 4.10 *Let* $\mathsf{F} = (W, R)$ *be a locally* 2*-component frame. Then for any* $w \in W$ *the following conditions are equivalent:*

(1) $\mathsf{F}, w \vDash \textsc{AxCov}$*;*

(2) if $\#_{\mathsf{F}}(w) = 2$ *and* $V \in comp_{\mathsf{F}}(w)$*, then* $R(V) \supseteq R^2(w) - R(w)$*.*

Proposition 4.11 *If frames* F, G *are locally one-component, then* $\mathsf{F} \curlyvee \mathsf{G} \vDash \textsc{AxCov}$.

Proof. Let $\mathsf{F} \times \mathsf{G} = (W, R_1, R_2)$. If $\#_{\mathsf{F}\curlyvee\mathsf{G}}(w) = 2$, then $comp_{\mathsf{F}} = \{R_1(w), R_2(w)\}$. By Proposition 4.10, $\mathsf{F} \curlyvee \mathsf{G} \vDash \textsc{AxCov}$. □

Proposition 4.12 *Let* F *be a 2-transitive frame with a root* w *such that* $\#_{\mathsf{F}}(w) = 2$ *and* $\mathsf{F}, w \vDash \textsc{AxCov}$*. Then for any valuation* θ *we have:* $(\mathsf{F}, \theta), w \vDash comp(\mathsf{p}, \mathsf{q})$ *iff* $comp_{\mathsf{F}}(w) = \{\theta(\mathsf{p}), \theta(\mathsf{q})\}$*.*

Proof. Follows from Proposition 4.10. □

5 Simulation of two dimensions

In this section we define relations $\hat{R}^{\mathsf{M}}_h$, $\hat{R}^{\mathsf{M}}_v$ and operators $\Diamond^{\curlyvee}_v$, $\Diamond^{\curlyvee}_h$ that play the same role as $\bar{R}^{\mathsf{M}}_h$, $\bar{R}^{\mathsf{M}}_v$, $\Diamond_v$, $\Diamond_h$, but ‘work’ in the unimodal case.

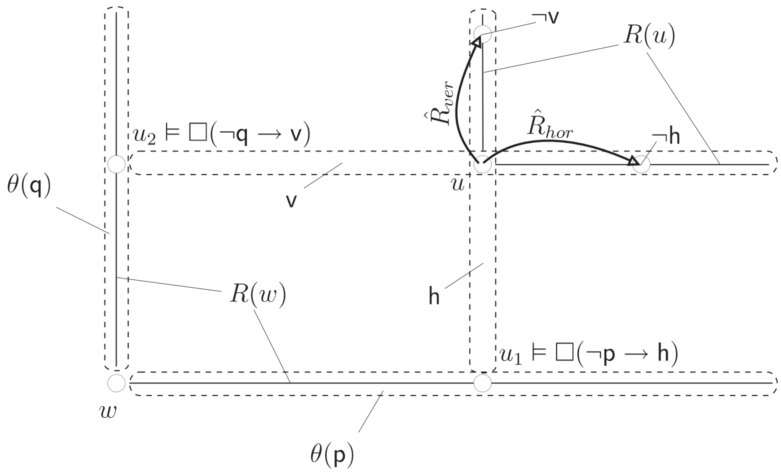


Fig. 4.

The main idea is that if $comp_{\mathsf{F}}(w)$=2, then we can split $R(w)$ into two parts (via the formula COMP(p, q)), and then express two 'directions' by unimodal operators. We show that under some additional restrictions these 'directions' are [K4,K4]-relations.

The formal definition of $\bar{R}_h^{\mathsf{M}}$, $\bar{R}_v^{\mathsf{M}}$ and $\Diamond_v^{\curlyvee}$, $\Diamond_h^{\curlyvee}$ is rather tedious, so first we illustrate the idea with the following example.

Example 5.1 (Fig. 4) Let F_1 and F_2 be rooted strict linear orders, $\mathsf{F} = \mathsf{F}_1 \curlyvee \mathsf{F}_2 = (W, R)$, $\mathsf{F}_1 \times \mathsf{F}_2 = (W, R_1, R_2)$. Let w be the root of F, θ be a valuation on W such that $\theta(\mathsf{p}) = R_1(w)$, $\theta(\mathsf{q}) = R_2(w)$. Suppose

$$(\mathsf{F}, \theta), w \models \Box_{\mathsf{p}}(\Box_{\neg\mathsf{p}}\mathsf{h} \vee \Box_{\neg\mathsf{p}}\neg\mathsf{h}) \wedge \Box_{\mathsf{q}}(\Box_{\neg\mathsf{q}}\mathsf{v} \vee \Box_{\neg\mathsf{q}}\neg\mathsf{v}),$$

or, equivalently, for any $u_1 \in R_1(w), u_2 \in R_2(w)$ we have

$$R_2(u_1) \cap \theta(\mathsf{h}) = \varnothing \text{ or } R_2(u_1) \subseteq \theta(\mathsf{h}), \quad R_1(u_2) \cap \theta(\mathsf{v}) = \varnothing \text{ or } R_1(u_2) \subseteq \theta(\mathsf{v}).$$

In this case we can define 'horizontal' and 'vertical' relations in terms of R and θ in the following way: for any $u \notin R^{=}(w)$, put

$$u\hat{R}_{hor}u' \text{ iff } uRu' \& (u \in \theta(\mathsf{h}) \Leftrightarrow u' \notin \theta(\mathsf{h})), \quad u\hat{R}_{ver}u' \text{ iff } uRu' \& (u \in \theta(\mathsf{v}) \Leftrightarrow u' \notin \theta(\mathsf{v})).$$

Now consider the 2-model $\mathsf{M} = (\mathsf{F}_1 \times \mathsf{F}_2, \theta)$ and observe that

$$(u, u') \in \hat{R}_{hor} \Leftrightarrow (u, u') \in \bar{R}_{h,0}^{\mathsf{M}} \text{ and } (u, u') \in \hat{R}_{ver} \Leftrightarrow (u, u') \in \bar{R}_{v,0}^{\mathsf{M}}$$

for any $u \notin R^{=}(w)$. Due to this observation, it is possible to define unimodal formulas which express $\bar{R}_v^{\mathsf{M}}$, $\bar{R}_h^{\mathsf{M}}$, thus to express $[\mathrm{K4}, \mathrm{K4}]$-relations in (F, θ).

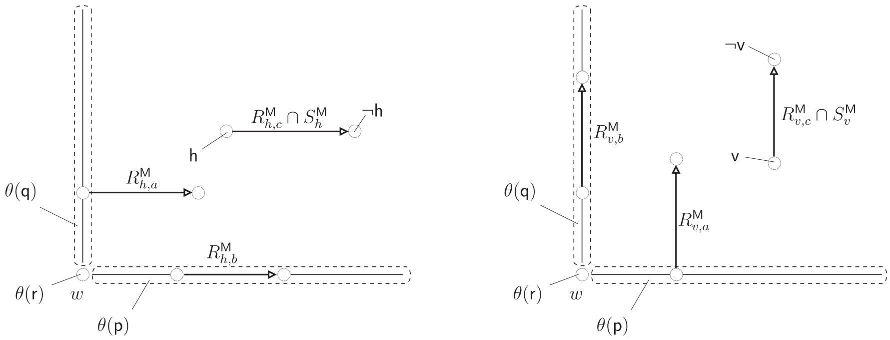


Fig. 5.

Definition 5.2 For a model $\mathsf{M} = (W, R, \theta)$, put

$$S^{\mathsf{M}}_h = \{(u,w) \mid uRw\&(u \in \theta(\mathsf{h}) \Leftrightarrow w \notin \theta(\mathsf{h}))\},$$
$$S^{\mathsf{M}}_v = \{(u,w) \mid uRw\&(u \in \theta(\mathsf{v}) \Leftrightarrow w \notin \theta(\mathsf{v}))\}.$$

Proposition 5.3 *For a unimodal* M*, if* $\psi \overset{\mathsf{M}}{\rightarrow} \widetilde{R}$*, then*

$$\bigwedge_{\varepsilon=0,1} (\mathsf{h}^\varepsilon \rightarrow \psi(\mathsf{s} \wedge \neg\mathsf{h}^\varepsilon)) \ \overset{\mathsf{M}}{\rightarrow}\ \widetilde{R} \cap S^{\mathsf{M}}_h, \quad \bigwedge_{\varepsilon=0,1} (\mathsf{v}^\varepsilon \rightarrow \psi(\mathsf{s} \wedge \neg\mathsf{v}^\varepsilon)) \ \overset{\mathsf{M}}{\rightarrow}\ \widetilde{R} \cap S^{\mathsf{M}}_v.$$

Proof. By a straightforward argument using Proposition 3.2. □

Definition 5.4 For a model $\mathsf{M} = (W, R, \theta)$, put

$$uR^{\mathsf{M}}_{h,a}v \Leftrightarrow uRv\&u \in \theta(\mathsf{q}) \cup \theta(\mathsf{r})\&v \notin \theta(\mathsf{q}), \quad uR^{\mathsf{M}}_{v,a}v \Leftrightarrow uRv\&u \in \theta(\mathsf{p}) \cup \theta(\mathsf{r})\&v \notin \theta(\mathsf{p}),$$
$$uR^{\mathsf{M}}_{h,b}v \Leftrightarrow uRv\&u, v \in \theta(\mathsf{p}), \quad uR^{\mathsf{M}}_{v,b}v \Leftrightarrow uRv\&u, v \in \theta(\mathsf{q}),$$
$$uR^{\mathsf{M}}_{h,c}v \Leftrightarrow uRv\&u \notin \theta(\mathsf{p}) \cup \theta(\mathsf{q}) \cup \theta(\mathsf{r}), \quad R^{\mathsf{M}}_{v,c} = R^{\mathsf{M}}_{h,c},$$
$$\hat{R}^{\mathsf{M}}_{h,0} = (R^{\mathsf{M}}_{h,a} \cup R^{\mathsf{M}}_{h,b} \cup R^{\mathsf{M}}_{h,c}) \cap S^{\mathsf{M}}_h, \quad \hat{R}^{\mathsf{M}}_{v,0} = (R^{\mathsf{M}}_{v,a} \cup R^{\mathsf{M}}_{v,b} \cup R^{\mathsf{M}}_{v,c}) \cap S^{\mathsf{M}}_v,$$
$$\hat{R}^{\mathsf{M}}_h = \hat{R}^{\mathsf{M}}_{h,0} \cup \left(\hat{R}^{\mathsf{M}}_{h,0}\right)^2, \quad \hat{R}^{\mathsf{M}}_v = \hat{R}^{\mathsf{M}}_{v,0} \cup \left(\hat{R}^{\mathsf{M}}_{v,0}\right)^2.$$

In Fig. 5 these relations are shown for the model described in Example 5.1 where also $\theta(\mathsf{r}) = \{w\}$ is assumed. Note that $R^{\mathsf{M}}_{h,c} \cap S^{\mathsf{M}}_h = \hat{R}_{hor}$, $R^{\mathsf{M}}_{v,c} \cap S^{\mathsf{M}}_v = \hat{R}_{ver}$, so $\hat{R}^{\mathsf{M}}_{h,0} \subseteq R_1$, and $\hat{R}^{\mathsf{M}}_{v,0} \subseteq R_2$; moreover, as we will show later, $\hat{R}^{\mathsf{M}}_h = \bar{R}^{\mathsf{M}}_h$ and $\hat{R}^{\mathsf{M}}_v = \bar{R}^{\mathsf{M}}_v$. It is not hard to see that these relations can be expressed in M by unimodal formulas. For this, we need several simple formulas, namely:

$$\psi^{\curlyvee}_{h,a} = \mathsf{q} \vee \mathsf{r} \to \Diamond(\neg \mathsf{q} \wedge \mathsf{s}), \qquad \psi^{\curlyvee}_{v,a} = \mathsf{p} \vee \mathsf{r} \to \Diamond(\neg \mathsf{p} \wedge \mathsf{s}),$$

$$\psi^{\curlyvee}_{h,b} = \mathsf{p} \to \Diamond(\mathsf{p} \wedge \mathsf{s}), \qquad \psi^{\curlyvee}_{v,b} = \mathsf{q} \to \Diamond(\mathsf{q} \wedge \mathsf{s}),$$

$$\psi^{\curlyvee}_{h,c} = \neg(\mathsf{p} \vee \mathsf{q} \vee \mathsf{r}) \to \Diamond \mathsf{s}, \qquad \psi^{\curlyvee}_{v,c} = \neg(\mathsf{p} \vee \mathsf{q} \vee \mathsf{r}) \to \Diamond \mathsf{s},$$

$$\psi^{\curlyvee}_{h,0} = \bigwedge_{\varepsilon=0,1} \left(\mathsf{h}^{\varepsilon} \to [(\mathsf{s} \wedge \neg \mathsf{h}^{\varepsilon})/\mathsf{s}](\psi^{\curlyvee}_{h,a} \wedge \psi^{\curlyvee}_{h,b} \wedge \psi^{\curlyvee}_{h,c})\right),$$

$$\psi^{\curlyvee}_{v,0} = \bigwedge_{\varepsilon=0,1} \left(\mathsf{v}^{\varepsilon} \to [(\mathsf{s} \wedge \neg \mathsf{v}^{\varepsilon})/\mathsf{s}](\psi^{\curlyvee}_{v,a} \wedge \psi^{\curlyvee}_{v,b} \wedge \psi^{\curlyvee}_{v,c})\right),$$

$$\psi^{\curlyvee}_{h} = \psi^{\curlyvee}_{h,0} \vee \psi^{\curlyvee}_{h,0}(\psi^{\curlyvee}_{h,0}), \qquad \psi^{\curlyvee}_{v} = \psi^{\curlyvee}_{v,0} \vee \psi^{\curlyvee}_{v,0}(\psi^{\curlyvee}_{v,0}).$$

The only subtle case is for the operators $\psi^{\curlyvee}_{h,0}$, $\psi^{\curlyvee}_{v,0}$: here we use Proposition 5.3.

Lemma 5.5 *Consider a model* $\mathsf{M} = (W, R, \theta)$ *such that the sets* $\theta(\mathsf{p})$, $\theta(\mathsf{q})$, $\theta(\mathsf{r})$ *are pairwise disjoint. Then*

$$\psi^{\curlyvee}_{h} \overset{\mathsf{M}}{\rightharpoondown} \hat{R}^{\mathsf{M}}_{h}, \quad \psi^{\curlyvee}_{v} \overset{\mathsf{M}}{\rightharpoondown} \hat{R}^{\mathsf{M}}_{v}.$$

Proof. By a straightforward argument,

$$\psi^{\curlyvee}_{h,a} \wedge \psi^{\curlyvee}_{h,b} \wedge \psi^{\curlyvee}_{h,c} \overset{\mathsf{M}}{\rightharpoondown} R^{\mathsf{M}}_{h,a} \cup R^{\mathsf{M}}_{h,b} \cup R^{\mathsf{M}}_{h,c}, \quad \psi^{\curlyvee}_{v,a} \wedge \psi^{\curlyvee}_{v,b} \wedge \psi^{\curlyvee}_{v,c} \overset{\mathsf{M}}{\rightharpoondown} R^{\mathsf{M}}_{v,a} \cup R^{\mathsf{M}}_{v,b} \cup R^{\mathsf{M}}_{v,c}.$$

By Proposition 5.3,

$$\psi^{\curlyvee}_{h,0} \overset{\mathsf{M}}{\rightharpoondown} \hat{R}^{\mathsf{M}}_{h,0}, \quad \psi^{\curlyvee}_{v,0} \overset{\mathsf{M}}{\rightharpoondown} \hat{R}^{\mathsf{M}}_{v,0},$$

thus $\psi^{\curlyvee}_{h} \overset{\mathsf{M}}{\rightharpoondown} \hat{R}^{\mathsf{M}}_{h}$ and $\psi^{\curlyvee}_{v} \overset{\mathsf{M}}{\rightharpoondown} \hat{R}^{\mathsf{M}}_{v}$. ◻

Put $\Diamond^{\curlyvee}_{h}\varphi = \psi^{\curlyvee}_{h}(\varphi)$, $\Diamond^{\curlyvee}_{v}\varphi = \psi^{\curlyvee}_{v}(\varphi)$, $\Box^{\curlyvee}_{h}\varphi = \neg\psi^{\curlyvee}_{h}(\neg\varphi)$, $\Box^{\curlyvee}_{v}\varphi = \neg\psi^{\curlyvee}_{v}(\neg\varphi)$.

5.1 Undecidability

The following formula is a unimodal analogue of the formula ψ_{hv}:

$$\psi^{\curlyvee}_{hv} = \mathsf{r} \wedge \neg\Diamond\mathsf{r} \wedge \neg\Diamond\Diamond\mathsf{r} \wedge \left(\Box_{\mathsf{p}}(\Box_{\neg\mathsf{p}}\mathsf{h} \vee \Box_{\neg\mathsf{p}}\neg\mathsf{h}) \wedge \Box_{\mathsf{q}}(\Box_{\neg\mathsf{q}}\mathsf{v} \vee \Box_{\neg\mathsf{q}}\neg\mathsf{v})\right).$$

Using it we express the relations $\bar{R}^{\mathsf{M}}_{h}$, $\bar{R}^{\mathsf{M}}_{v}$ in 1-models based on $\curlyvee$-products of transitive locally one-component frames.

Lemma 5.6 *Let* $\mathsf{F}_1, \mathsf{F}_2$ *be transitive locally one-component frames,* $\mathsf{F} = \mathsf{F}_1 \curlyvee \mathsf{F}_2$, $\mathsf{G} = \mathsf{F}_1 \times \mathsf{F}_2 = (W, R_1, R_2)$*,* θ *be a valuation on* W *such that*

$$\theta(\mathsf{p}) = R_1(w),\ \theta(\mathsf{q}) = R_2(w),\ \theta(\mathsf{r}) = \{w\}. \tag{3}$$

If F *has the irreflexive root* w*, then the following holds.*

(i) $(\mathsf{F}, \theta), w \vDash \psi^{\curlyvee}_{hv}$ *iff* $(\mathsf{G}, \theta), w \vDash \psi_{hv}$.

(ii) If $(\mathsf{F}, \theta), w \vDash \psi^{\curlyvee}_{hv}$*, then* $\hat{R}^{(\mathsf{F},\theta)}_{h} = \bar{R}^{(\mathsf{G},\theta)}_{h}$, $\hat{R}^{(\mathsf{F},\theta)}_{v} = \bar{R}^{(\mathsf{G},\theta)}_{v}$.

(iii) If $(\mathsf{F},\theta), w \vDash \psi^{\curlyvee}_{hv}$, $\varphi \in ML_2$, $u \in W$, *then*

$$(\mathsf{F},\theta), u \vDash [\varphi]_{(\psi^{\curlyvee}_h,\psi^{\curlyvee}_v)} \;\Leftrightarrow\; (\mathsf{G},\theta), u \vDash [\varphi]_{(\psi_h,\psi_v)}.$$

Proof. Put $R = R_1 \cup R_2$, $\mathsf{M} = (\mathsf{F},\theta)$, $\mathsf{N} = (\mathsf{G},\theta)$, $w = (x_0, y_0)$. Note that

$$R_1(R_2(w)) \cap R_1^{=}(w) = R_1(R_2(w)) \cap R_2^{=}(w) = \varnothing. \tag{4}$$

Indeed, for $u = (x, y) \in R_1(R_2(w))$, if $u \in R_1^{=}(w)$, then $y = y_0$ and y_0 is reflexive in F_2, and if $u \in R_2^{=}(w)$, then $x = x_0$ and x_0 is reflexive if F_1; since $\#_{\mathsf{F}}(w) = 2$, then by Proposition 4.8 x_0 is irreflexive in F_1 and y_0 is irreflexive in F_2, that proves (4).

(i) Suppose $\mathsf{M}, w \vDash \psi^{\curlyvee}_{hv}$.

Consider $u = (x, y) \in R_1(R_2(w))$.

Let $\mathsf{M}, u \vDash \mathsf{h}^{\varepsilon} \vee \Diamond_2 \mathsf{h}^{\varepsilon}$, $v \in R_2(u)$ for some $v \in R_2(w)$, $\varepsilon \in \{0,1\}$. Since $wR_1(x, y_0)R_2u$, then $(x, y_0) \in \theta(\mathsf{p})$, and $\mathsf{M}, (x, y_0) \vDash \Box_{\neg\mathsf{p}}\mathsf{h} \vee \Box_{\neg\mathsf{p}}\neg\mathsf{h}$. Due to (4), $u, v \notin \theta(\mathsf{p})$, so $\mathsf{M}, u \vDash \Box_{\neg\mathsf{p}}\mathsf{h}^{\varepsilon}$ and $\mathsf{N}, v \vDash \mathsf{h}^{\varepsilon}$. Thus $\mathsf{M}, u \vDash \mathsf{h}^{\varepsilon} \vee \Diamond_2\mathsf{h}^{\varepsilon} \to \Box_2\mathsf{h}^{\varepsilon}$.

Similarly, $\mathsf{M}, u \vDash \mathsf{v}^{\varepsilon} \vee \Diamond_1\mathsf{v}^{\varepsilon} \to \Box_1\mathsf{v}^{\varepsilon}$, so $\mathsf{N}, w \vDash \psi_{hv}$.

Suppose $\mathsf{N}, w \vDash \psi_{hv}$.

Let us show that $\mathsf{M}, w \vDash \bigwedge\limits_{\varepsilon=0,1} (\Box_{\mathsf{p}}(\Diamond_{\neg\mathsf{p}}\mathsf{h}^{\varepsilon} \to \Box_{\neg\mathsf{p}}\mathsf{h}^{\varepsilon}))$.

Suppose $u \in R(x)$, $u \in \theta(\mathsf{p})$. Then wR_1u, $u = (x, y_0)$ for some x. Put

$$Y_0 = R_2(u) - \theta(\mathsf{h}),\ Y_1 = R_2(u) \cap \theta(\mathsf{h}).$$

Since $\mathsf{N}, (x, y) \vDash \mathsf{h} \vee \Diamond_2\mathsf{h} \to \Box_2\mathsf{h}$ for any $(x, y) \in Y_1$, then $R_2(Y_1) \cap Y_0 = \varnothing$. Similarly, $R_2(Y_0) \cap Y_1 = \varnothing$. Since F_2 is locally one-component, $Y_0 = \varnothing$ or $Y_1 = \varnothing$. Suppose $\mathsf{M}, u \vDash \Diamond_{\neg\mathsf{p}}\mathsf{h}^{\varepsilon}$ for some $\varepsilon \in \{0,1\}$, and let $v \in R(u) - \theta(\mathsf{p})$. Then $\mathsf{M}, u' \vDash \mathsf{h}^{\varepsilon}$ for some $u' \in R(u) - \theta(\mathsf{p})$. It follows that $u', v \notin R_1(u)$, so $u', v \in R_2(u)$. Thus $Y_{\varepsilon} \neq \varnothing$ and $v \in Y_{\varepsilon}$. It follows that $\mathsf{M}, v \vDash \mathsf{h}^{\varepsilon}$, so $\mathsf{M}, u \vDash \Diamond_{\neg\mathsf{p}}\mathsf{h}^{\varepsilon} \to \Box_{\neg\mathsf{p}}\mathsf{h}^{\varepsilon}$.

Similarly, $\mathsf{M}, w \vDash \Box_{\mathsf{q}}(\Box_{\neg\mathsf{q}}\mathsf{v} \vee \Box_{\neg\mathsf{q}}\neg\mathsf{v})$.

Since w is irreflexive in F, $\mathsf{M}, w \vDash \neg\Diamond\mathsf{r}$, and due to (4), $\mathsf{M}, w \vDash \neg\Diamond\Diamond\mathsf{r}$. It follows that $\mathsf{M}, w \vDash \psi^{\curlyvee}_{hv}$.

(ii) Let us show that $\hat{R}^{\mathsf{M}}_{h,0} = \bar{R}^{\mathsf{N}}_{h,0}$. Note that if $u\hat{R}^{\mathsf{M}}_{h,0}v$ or $u\bar{R}^{\mathsf{N}}_{h,0}v$, then

$$uRv \text{ and } (u \in \theta(\mathsf{h}) \Leftrightarrow v \notin \theta(\mathsf{h})). \tag{5}$$

Suppose (5) holds and consider the following cases.
a). $u \in R_2^{=}(w)$ (equivalently, $u \in \theta(\mathsf{q}) \cup \theta(\mathsf{r})$).
In this case, uR_1v iff $v \notin \theta(\mathsf{q})$, so $u\bar{R}^{\mathsf{N}}_{h,0}v$ iff $uR^{\mathsf{M}}_{h,a}v$.
b). $u \in R_1(w)$ (equivalently, $u \in \theta(\mathsf{p})$).
In this case, uR_1v iff $v \in \theta(\mathsf{p})$, so $u\bar{R}^{\mathsf{M}}_{h,0}v$ iff $uR^{\mathsf{M}}_{h,b}v$.
c). $u \in R^2(w) - R^{=}(w)$ (equivalently, $u \notin \theta(\mathsf{p}) \cup \theta(\mathsf{q}) \cup \theta(\mathsf{r})$).
In this case we have $uR^{\mathsf{M}}_{h,c}v$. Let us show that $u\bar{R}^{\mathsf{M}}_{h,0}v$. Due to (5), $u\bar{R}^{\mathsf{M}}_{h,0}v$ iff uR_1v. Since $\mathsf{N}, w \vDash \psi_{hv}$, $\mathsf{N}, u \vDash \bigwedge\limits_{\varepsilon=0,1} (\mathsf{h}^{\varepsilon} \vee \Diamond_2\mathsf{h}^{\varepsilon} \to \Box_2\mathsf{h}^{\varepsilon})$. It follows that if uR_2v then $(u \in \theta(\mathsf{h}) \Leftrightarrow v \in \theta(\mathsf{h}))$, which contradicts (5). Thus uR_1v, and so $u\bar{R}^{\mathsf{M}}_{h,0}v$.

It follows that $\hat{R}^{\mathsf{M}}_{h,0} = \bar{R}^{\mathsf{N}}_{h,0}$. Similarly, $\hat{R}^{\mathsf{M}}_{v,0} = \bar{R}^{\mathsf{N}}_{v,0}$. Thus $\hat{R}^{\mathsf{M}}_{h} = \bar{R}^{\mathsf{N}}_{h}$ and $\hat{R}^{\mathsf{M}}_{v} = \bar{R}^{\mathsf{N}}_{v}$.

(iii) Follows from (2), Lemma 5.5, and Lemma 3.6. □

The above lemma allows to express [K4, K4]-relations in models: to apply the lemma, we need the condition (3). The following key lemma shows how to obtain this result for frames.

Lemma 5.7 *Let* $\mathsf{F}_1, \mathsf{F}_2$ *be transitive locally one-component frames,* (x, y) *be an irreflexive point in* $\mathsf{F}_1 \curlyvee \mathsf{F}_2$. *Let* $\varphi \in ML_2$, $PV(\varphi) \cap \{\mathsf{p}, \mathsf{q}, \mathsf{r}\} = \varnothing$. *Then* $\textsc{comp}(\mathsf{p}, \mathsf{q}) \wedge \psi^{\curlyvee}_{hv} \wedge [\varphi]_{(\psi^{\curlyvee}_h, \psi^{\curlyvee}_v)}$ *is satisfiable at* (x, y) *in* $\mathsf{F}_1 \curlyvee \mathsf{F}_2$ *iff* $\psi_{hv} \wedge [\varphi]_{(\psi_h, \psi_v)}$ *is satisfiable at* (x, y) *in* $\mathsf{F}_1 \times \mathsf{F}_2$ *or at* (y, x) *in* $\mathsf{F}_2 \times \mathsf{F}_1$.

Proof. Let F denote $(\mathsf{F}_1 \curlyvee \mathsf{F}_2)^{(x,y)}$, $(W, R_1, R_2) = (\mathsf{F}_1 \times \mathsf{F}_2)^{(x,y)}$.

Since (x, y) is irreflexive, $\#_{\mathsf{F}}(x, y) = 2$ (Proposition 4.8).

For $V \subseteq W$, put $V^* = \{(y', x') \mid (x', y') \in V\}$. For a valuation θ on W, let $\theta^*(\mathsf{t}) = (\theta(\mathsf{t}))^*$ for all $\mathsf{t} \in PV$. Trivially, $(\mathsf{F}_1 \curlyvee \mathsf{F}_2, \theta), (x', y') \vDash \psi \Leftrightarrow (\mathsf{F}_2 \curlyvee \mathsf{F}_1, \theta^*), (y', x') \vDash \psi$ for any $\psi \in ML_1$, $(x', y') \in W$.

Suppose $(\mathsf{F}, \theta), (x, y) \vDash \textsc{comp}(\mathsf{p}, \mathsf{q}) \wedge \psi^{\curlyvee}_{hv} \wedge [\varphi]_{(\psi^{\curlyvee}_h, \psi^{\curlyvee}_v)}$. Since $\#_{\mathsf{F}}(x, y) = 2$, $comp_{\mathsf{F}}(x, y) = \{\theta(\mathsf{p}), \theta(\mathsf{q})\}$. It follows that

$$R_1(x, y) = \theta(\mathsf{p}) \ \&\ R_2(x, y) = \theta(\mathsf{q}) \ \&\ \{(x, y)\} = \theta(\mathsf{r}) \quad \text{or}$$
$$R_2(x, y) = \theta(\mathsf{p}) \ \&\ R_1(x, y) = \theta(\mathsf{q}) \ \&\ \{(x, y)\} = \theta(\mathsf{r}).$$

By Lemma 5.6, in the former case $(\mathsf{F}_1 \times \mathsf{F}_2, \theta), (x, y) \vDash \psi_{hv} \wedge [\varphi]_{(\psi_h, \psi_v)}$, and in the latter case $(\mathsf{F}_2 \times \mathsf{F}_1, \theta^*), (y, x) \vDash \psi_{hv} \wedge [\varphi]_{(\psi_h, \psi_v)}$.

If $(\mathsf{F}_1 \times \mathsf{F}_2, \theta), (x, y) \vDash \psi_{hv} \wedge [\varphi]_{(\psi_h, \psi_v)}$, put

$$\eta(\mathsf{p}) = R_1(x, y),\ \eta(\mathsf{q}) = R_2(x, y),\ \eta(\mathsf{r}) = \{(x, y)\},\ \eta(\mathsf{t}) = \theta(\mathsf{t}) \text{ for } \mathsf{t} \notin \{\mathsf{p}, \mathsf{q}, \mathsf{r}\};$$

if $(\mathsf{F}_2 \times \mathsf{F}_1, \theta), (y, x) \vDash \psi_{hv} \wedge [\varphi]_{(\psi_h, \psi_v)}$, let η be a valuation on W^* such that

$$\eta(\mathsf{p}) = (R_2(x, y))^*,\ \eta(\mathsf{q}) = (R_1(x, y))^*,\ \eta(\mathsf{r}) = \{(y, x)\},\ \eta(\mathsf{t}) = \theta(\mathsf{t}) \text{ for } \mathsf{t} \notin \{\mathsf{p}, \mathsf{q}, \mathsf{r}\}.$$

Since $PV([\varphi]_{(\psi_h, \psi_v)}, \psi_{hv}) \cap \{\mathsf{p}, \mathsf{q}, \mathsf{r}\} = \varnothing$, then $(\mathsf{F}_1 \times \mathsf{F}_2, \eta), (x, y) \vDash \psi_{hv} \wedge [\varphi]_{(\psi_h, \psi_v)}$ or $(\mathsf{F}_2 \times \mathsf{F}_1, \eta), (y, x) \vDash \psi_{hv} \wedge [\varphi]_{(\psi_h, \psi_v)}$. Moreover, if the former case $(\mathsf{F}_1 \times \mathsf{F}_2, \eta), (x, y) \vDash \textsc{comp}(\mathsf{p}, \mathsf{q})$, and in the latter case $(\mathsf{F}_2 \times \mathsf{F}_1, \eta), (y, x) \vDash \textsc{comp}(\mathsf{p}, \mathsf{q})$. By Lemma 5.6, $\textsc{comp}(\mathsf{p}, \mathsf{q}) \wedge \psi^{\curlyvee}_{hv} \wedge [\varphi]_{(\psi^{\curlyvee}_h, \psi^{\curlyvee}_v)}$ is satisfiable at (x, y) in $\mathsf{F}_1 \curlyvee \mathsf{F}_2$ or at (y, x) in $\mathsf{F}_2 \curlyvee \mathsf{F}_1$. To finish the proof, note that $\mathsf{F}_1 \curlyvee \mathsf{F}_2$ and $\mathsf{F}_2 \curlyvee \mathsf{F}_1$ are isomorphic. □

The above lemma allows us to formulate undecidability results for unimodal frames.

Definition 5.8 A transitive locally one-component frame $\mathsf{F} = (W, R)$ is called *an axis-frame*, if there exists an irreflexive point x in F such that F^x contains an infinite descending chain of distinct points, i.e., there exists a sequence $\{x_i\}_{i>0}$ such that $x_{i+1} R x_i$, $x_i \neq x_{i+1}$ and $x R x_i$ for all $i > 0$; x is called *an origin of* F.

Theorem 5.9 *If a class $\mathcal{F}$ of transitive locally one-component frames contains an axis-frame, then $\mathbf{L}(\mathcal{F} \curlyvee \mathcal{F})$ is undecidable.*

Proof. In [7], it was shown that the $\omega \times \omega$-tiling problem is reducible to the [K4, K4]-satisfiability problem. More precisely, there was described a procedure which for a given tile Θ provides φ^Θ with the following properties:

(i) if Θ tiles $\omega \times \omega$ and F is a transitive 1-frame with a root x containing an infinite descending chain, then $\psi_{hv} \wedge [\varphi^\Theta]_{(\psi_h,\psi_v)}$ is satisfiable at (x, x) in $\mathsf{F} \times \mathsf{F}$;

(ii) if $\psi_{hv} \wedge [\varphi^\Theta]_{(\psi_h,\psi_v)}$ is [K4, K4]-satisfiable, then Θ tiles $\omega \times \omega$. [2]

Without any loss of generality we may assume that $PV(\varphi^\Theta) \cap \{\mathsf{p}, \mathsf{q}, \mathsf{r}\} = \varnothing$.

The statement of the theorem follows from the following fact:

$$\Theta \text{ tiles } \omega \times \omega \text{ iff } \textsc{comp}(\mathsf{p}, \mathsf{q}) \wedge \psi^\curlyvee_{hv} \wedge [\varphi^\Theta]_{(\psi^\curlyvee_h,\psi^\curlyvee_v)} \text{ is } \mathcal{F} \curlyvee \mathcal{F}\text{-satisfiable.}$$

To prove it, consider an axis-frame $\mathsf{F} \in \mathcal{F}$ with an origin point x. If Θ tiles $\omega \times \omega$, then $\psi_{hv} \wedge [\varphi^\Theta]_{(\psi_h,\psi_v)}$ is satisfiable at (x, x) in $(\mathsf{F} \times \mathsf{F})^{(x,x)}$, and by Lemma 5.7, $\textsc{comp}(\mathsf{p}, \mathsf{q}) \wedge \psi^\curlyvee_{hv} \wedge [\varphi^\Theta]_{(\psi^\curlyvee_h,\psi^\curlyvee_v)}$ is satisfiable at (x, x) in $\mathsf{F} \curlyvee \mathsf{F}$.

Conversely, suppose $\textsc{comp}(\mathsf{p}, \mathsf{q}) \wedge \psi^\curlyvee_{hv} \wedge [\varphi^\Theta]_{(\psi^\curlyvee_h,\psi^\curlyvee_v)}$ is satisfiable at a point (x, y) in a frame $\mathsf{F}_1 \curlyvee \mathsf{F}_2$ for some $\mathsf{F}_1, \mathsf{F}_2 \in \mathcal{F}$. Then $\#_{\mathsf{F}_1 \curlyvee \mathsf{F}_2}(x, y) = 2$ due to Proposition 4.7. By Lemma 5.7, $\psi_{hv} \wedge [\varphi^\Theta]_{(\psi_h,\psi_v)}$ is satisfiable at (x, y) in $\mathsf{F}_1 \times \mathsf{F}_2$ or at (y, x) in $\mathsf{F}_2 \times \mathsf{F}_1$. Thus Θ tiles $\omega \times \omega$. $\square$

Example 5.10 Clearly, if $\mathsf{F} = (W, R)$ is a strict linear order containing a point x and an infinite descending chain $y_1 R^{-1} y_2 R^{-1} \ldots R^{-1} x$, then F is an axis frame. Thus the satisfiability problem for $\mathsf{F} \curlyvee \mathsf{F}$ is undecidable. In particular, the satisfiability problems for $(\mathbb{R}, <) \curlyvee (\mathbb{R}, <)$ and $(\mathbb{Q}, <) \curlyvee (\mathbb{Q}, <)$ are undecidable.

5.2 Lack of the finite model property

In the previous subsection we 'encoded' two dimensions in semantically defined frames – namely, in $\curlyvee$-products. To prove the lack of finite model property, we have to define such frames axiomatically.

For a formula φ, let $\Box^{\leq 2}\varphi$ abbreviate $\Box\Box\varphi \wedge \Box\varphi \wedge \varphi$.

Let $\mathrm{L}^\curlyvee_{min}$ be the minimal normal unimodal logic containing the formulas $\Diamond^3\mathsf{p} \to \Diamond^2\mathsf{p}$, $\textsc{AxComp}_2$, $\textsc{AxCov}$, and the following formulas:

$$\begin{aligned}
\textsc{AxTr}^\curlyvee_1 &= \textsc{comp}(\mathsf{p}, \mathsf{q}) \wedge \psi^\curlyvee_{hv} \to \Box^{\leq 2}(\Diamond^\curlyvee_h \Diamond^\curlyvee_h \mathsf{t} \to \Diamond^\curlyvee_h \mathsf{t}),\\
\textsc{AxTr}^\curlyvee_2 &= \textsc{comp}(\mathsf{p}, \mathsf{q}) \wedge \psi^\curlyvee_{hv} \to \Box^{\leq 2}(\Diamond^\curlyvee_v \Diamond^\curlyvee_v \mathsf{t} \to \Diamond^\curlyvee_v \mathsf{t}),\\
\textsc{AxCR}^\curlyvee &= \textsc{comp}(\mathsf{p}, \mathsf{q}) \wedge \psi^\curlyvee_{hv} \to \Box^{\leq 2}(\Diamond^\curlyvee_h \Box^\curlyvee_v \mathsf{t} \to \Box^\curlyvee_h \Diamond^\curlyvee_v \mathsf{t}),\\
\textsc{AxComm}^\curlyvee &= \textsc{comp}(\mathsf{p}, \mathsf{q}) \wedge \psi^\curlyvee_{hv} \to \Box^{\leq 2}(\Diamond^\curlyvee_h \Diamond^\curlyvee_v \mathsf{t} \leftrightarrow \Diamond^\curlyvee_v \Diamond^\curlyvee_h \mathsf{t}).
\end{aligned}$$

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[2] See the proof of Theorem 2 in [7], where $\psi_{hv} \wedge [\varphi^\Theta]_{(\psi_h,\psi_v)}$ is the conjunction of the formulas denoted by φ_∞, φ_{grid}, and φ_Θ.

Lemma 5.11 *If* $\mathsf{F}_1, \mathsf{F}_2$ *are unimodal transitive locally one-component frames, then* $\mathsf{F}_1 \curlyvee \mathsf{F}_2 \vDash \mathrm{L}^{\curlyvee}_{min}$.

Proof. Due to Propositions 4.5, 4.9, and 4.11,

$$\mathsf{F}_1 \curlyvee \mathsf{F}_2 \vDash \{\Diamond^3 \mathsf{p} \to \Diamond^2 \mathsf{p}, \textsc{AxComp}_2, \textsc{AxCov}\}.$$

Due to Proposition 2.1 and Lemmas 5.6, 3.10,

$$\mathsf{F}_1 \curlyvee \mathsf{F}_2 \vDash \{\textsc{AxTr}^{\curlyvee}_1, \textsc{AxTr}^{\curlyvee}_2, \textsc{AxCR}^{\curlyvee}, \textsc{AxComm}^{\curlyvee}\}.$$

□

Proposition 5.12 *Let* $\mathsf{F} = (W, R_1, R_2)$ *be a* $[\mathrm{K4}, \mathrm{K4}]$*-frame,* $\mathsf{M} = (\mathsf{F}, \theta)$*,* $\mathsf{G} = (W, \bar{R}^{\mathsf{M}}_h, \bar{R}^{\mathsf{M}}_v)$*. Then* $\bar{R}^{(\mathsf{G},\theta)}_h = \bar{R}^{\mathsf{M}}_h$*,* $\bar{R}^{(\mathsf{G},\theta)}_v = \bar{R}^{\mathsf{M}}_v$*.*

Proof. Put $\mathsf{N} = (\mathsf{G}, \theta)$.

$u\bar{R}^{\mathsf{M}}_{h,0}v$ iff $uR_1v\&(\mathsf{M}, u \vDash \mathsf{h} \Leftrightarrow \mathsf{M}, v \vDash \neg\mathsf{h})$ iff $u\bar{R}^{\mathsf{M}}_{h,0}v\&(\mathsf{N}, u \vDash \mathsf{h} \Leftrightarrow \mathsf{N}, v \vDash \neg\mathsf{h})$. It follows that $\bar{R}^{\mathsf{M}}_{h,0} = \bar{R}^{\mathsf{N}}_{h,0}$. Similarly, $\bar{R}^{\mathsf{M}}_{v,0} = \bar{R}^{\mathsf{N}}_{v,0}$. □

Lemma 5.13 *Let* $\mathsf{F} = (W, R) \vDash \mathrm{L}^{\curlyvee}_{min}$*, w be a root of* F*,* $\theta : PV \to \mathcal{P}(W)$*,* $\mathsf{M} = (\mathsf{F}, \theta)$*,* $\mathsf{G} = (W, \hat{R}^{\mathsf{M}}_h, \hat{R}^{\mathsf{M}}_v)$*,* $\mathsf{N} = (\mathsf{G}, \theta)$*. Suppose that* $\mathsf{M}, w \vDash \psi^{\curlyvee}_{hv} \wedge \textsc{comp}(\mathsf{p}, \mathsf{q})$*. Then we have:*

(i) $\psi^{\curlyvee}_h \overset{\mathsf{M}}{\rightharpoondown} \hat{R}^{\mathsf{M}}_h$, $\psi^{\curlyvee}_v \overset{\mathsf{M}}{\rightharpoondown} \hat{R}^{\mathsf{M}}_v$;

(ii) G *is a* $[\mathrm{K4}, \mathrm{K4}]$*-frame;*

(iii) $\mathsf{N}, w \vDash \psi_{hv}$*;*

(iv) if $\varphi \in ML_2$*,* $PV(\varphi) \cap \{\mathsf{p}, \mathsf{q}, \mathsf{r}\} = \varnothing$*, then for any* $u \in W$

$$\mathsf{M}, u \vDash [\varphi]_{(\psi^{\curlyvee}_h, \psi^{\curlyvee}_v)} \;\Leftrightarrow\; \mathsf{N}, u \vDash [\varphi]_{(\psi_h, \psi_v)}.$$

Proof. Put $V_1 = \theta(\mathsf{p})$, $V_2 = \theta(\mathsf{q})$. By Proposition 4.12, $comp(w) = \{V_1, V_2\}$. It follows that $\theta(\mathsf{p})$, $\theta(\mathsf{q})$, $\theta(\mathsf{r})$ are pairwise disjoint, so (i) follows from Lemma 5.5. (ii) follows from (i), Lemma 5.11, and Lemma 3.10.

Let us check (iii). Put $R_1 = \bar{R}^{\mathsf{N}}_h$, $R_2 = \bar{R}^{\mathsf{N}}_v$. It follows that if $u \in R_2(R_1(w))$, then $u \notin V_2$, and if $u \in R_1(R_2(w))$, then $u \notin V_1$. Due to the commutativity, we have

$$R_1(R_2(w)) \subseteq R^2(w) - R(w). \tag{6}$$

Let $u \in R_2(R_1(w))$, $\mathsf{N}, u \vDash \mathsf{h}^{\varepsilon} \vee \Diamond \mathsf{h}^{\varepsilon}$ for some $\varepsilon \in \{0, 1\}$. Then $u_0 R_2 u$ and $\mathsf{N}, u' \vDash \mathsf{h}^{\varepsilon}$ for some $u_0 \in V_1$, $u_1 \in R_2^{=}(u)$. Thus $\mathsf{M}, u_0 \vDash \Diamond_{\neg\mathsf{p}} \mathsf{h}^{\varepsilon} \to \Box_{\neg\mathsf{p}} \mathsf{h}^{\varepsilon}$. Suppose $v \in R_2(u)$. Since R_2 is transitive, $u_1, v \in R_2(R_1(w))$. Due to (6), $u_1, v \notin \theta(\mathsf{p})$. Recall that $R \supseteq R_2$, thus $\mathsf{M}, u_0 \vDash \Diamond_{\neg\mathsf{p}} \mathsf{h}^{\varepsilon}$, so $\mathsf{M}, u_0 \vDash \Box_{\neg\mathsf{p}} \mathsf{h}^{\varepsilon}$ and $\mathsf{N}, v \vDash \mathsf{h}^{\varepsilon}$. It follows that $\mathsf{N}, u \vDash \bigwedge\limits_{\varepsilon=0,1} (\mathsf{h}^{\varepsilon} \vee \Diamond_2 \mathsf{h}^{\varepsilon} \to \Box_2 \mathsf{h}^{\varepsilon})$.

Analogously, $\mathsf{N}, u \vDash \bigwedge\limits_{\varepsilon=0,1} (\mathsf{v}^{\varepsilon} \vee \Diamond_1 \mathsf{v}^{\varepsilon} \to \Box_1 \mathsf{v}^{\varepsilon})$ for any $u \in R_2(R_1(w))$, which implies (iii).

By Proposition 5.12, $\hat{R}^{\mathsf{M}}_h = \bar{R}^{\mathsf{N}}_h$, $\hat{R}^{\mathsf{M}}_v = \bar{R}^{\mathsf{N}}_v$. Due to (i) we have $\psi^{\curlyvee}_h \overset{\mathsf{M}}{\rightharpoondown} \bar{R}^{\mathsf{N}}_h$, $\psi^{\curlyvee}_v \overset{\mathsf{M}}{\rightharpoondown} \bar{R}^{\mathsf{N}}_v$. Recall that $\psi_h \overset{\mathsf{N}}{\rightharpoondown} \bar{R}^{\mathsf{N}}_h$, $\psi_v \overset{\mathsf{N}}{\rightharpoondown} \bar{R}^{\mathsf{N}}_v$. By Lemma 3.6 we obtain (iv). □

Theorem 5.14 *If a unimodal logic* L *contains* $\mathrm{L}^{\curlyvee}_{min}$ *and there exists an axis-frame* F *such that* $\mathsf{F} \curlyvee \mathsf{F} \vDash \mathrm{L}$*, then* L *has no finite model property.*

Proof. In [7], it was shown that there exists a formula $\varphi^{diag} \in ML_2$ such that

(i) if F is a transitive 1-frame with a root x containing an infinite descending chain, then $\psi_{hv} \wedge [\varphi^{diag}]_{(\psi_h,\psi_v)}$ is satisfiable at (x,x) in $\mathsf{F} \times \mathsf{F}$,

(ii) if G is a [K4, K4]-frame and $\psi_{hv} \wedge [\varphi^{diag}]_{(\psi_h,\psi_v)}$ is satisfiable in G, then G is infinite.[3]

Due to Lemma 5.7, $\mathrm{COMP}(\mathsf{p},\mathsf{q}) \wedge \psi^{\curlyvee}_{hv} \wedge [\varphi^{diag}]_{(\psi^{\curlyvee}_h,\psi^{\curlyvee}_v)}$ is satisfiable in $\mathsf{F} \curlyvee \mathsf{F}$.

On the other hand, if $\mathrm{COMP}(\mathsf{p},\mathsf{q}) \wedge \psi^{\curlyvee}_{hv} \wedge [\varphi^{diag}]_{(\psi^{\curlyvee}_h,\psi^{\curlyvee}_v)}$ is satisfiable in a finite L-frame G, then, by Lemma 5.13, $\psi_{hv} \wedge [\varphi^{diag}]_{(\psi_h,\psi_v)}$ is satisfiable in a finite [K4, K4]-frame, which is a contradiction. ☐

Corollary 5.15 *If a class* $\mathcal{F}$ *of transitive locally one-component frames contains an axis-frame, then* $\mathbf{L}(\mathcal{F} \curlyvee \mathcal{F})$ *does not have the finite model property.*

Example 5.16 The logics $\mathbf{L}((\mathbb{R},<) \curlyvee (\mathbb{R},<))$ and $\mathbf{L}((\mathbb{Q},<) \curlyvee (\mathbb{Q},<))$ does not have the finite model property.

6 $\langle \overline{B} \vee \overline{E} \rangle$-fragment of HS

In this section we consider modal logics, where modal operators are interpreted by relations between intervals. For known results on these logics see [3,2,4].

We show how Theorems 5.9 and 5.14 can be used to obtain negative results for logics of intervals.

For a (strict or non-strict) partial order $\mathsf{F} = (W, R)$, let $Ints(W)$ denote *the set of all (non-strict) intervals over* F: $Ints(W) = \{(a,b) \mid aR^{=}b\}$. For intervals (a,b), (c,d),

$$(a,b)R_{\langle B \rangle}(c,d) \text{ iff } a = c \ \wedge \ dRb;$$
$$(a,b)R_{\langle E \rangle}(c,d) \text{ iff } aRc \ \wedge \ b = d;$$
$$R_{\langle B \vee E \rangle} = R_{\langle B \rangle} \cup R_{\langle E \rangle}, \ \ R_{\langle \overline{B} \vee \overline{E} \rangle} = R^{-1}_{\langle B \vee E \rangle}.$$

For a partial order F, let $\mathbf{L}_{\langle \overline{B} \vee \overline{E} \rangle}(\mathsf{F})$ denote $\mathbf{L}(Ints(W), R_{\langle \overline{B} \vee \overline{E} \rangle})$. For a class F of partial orders, put $\mathbf{L}_{\langle \overline{B} \vee \overline{E} \rangle}(\mathsf{F}) = \bigcap_{\mathsf{F} \in \mathcal{F}} \mathbf{L}_{\langle \overline{B} \vee \overline{E} \rangle}(\mathsf{F})$.

Lemma 6.1 *Let* $\mathsf{F} = (W, R)$ *be a partial order,* $\mathsf{G} = (W, R^{-1})$*. Then* $\mathbf{L}_{\langle \overline{B} \vee \overline{E} \rangle}(\mathsf{F}) = \bigcap_{aR^{=}b} \mathbf{L}(\mathsf{G}^a \curlyvee \mathsf{F}^b)$.

Proof. The statement of the lemma is based on the following observation (see e.g. [10]): $(Ints(W), R_{\langle \overline{E} \rangle}, R_{\langle \overline{B} \rangle})^i = \mathsf{G}^a \times \mathsf{F}^b$ for any $i = (a,b) \in Ints(W)$; therefore,

$$(Ints(W), R_{\langle \overline{B} \vee \overline{E} \rangle})^i = \mathsf{G}^a \curlyvee \mathsf{F}^b. \qquad (7)$$

[3] In [7], $\psi_{hv} \wedge [\varphi^{diag}]_{(\psi_h,\psi_v)}$ is denoted by ψ_∞.

We have:

$$\mathbf{L}(Ints(W), R_{\langle \overline{B} \vee \overline{E} \rangle}) = \bigcap_{aR=b} (Ints(W), R_{\langle \overline{B} \vee \overline{E} \rangle})^{(a,b)} = \bigcap_{aR=b} \mathbf{L}(\mathsf{G}^a \curlyvee \mathsf{F}^b).$$

□

Theorem 6.2 *Let $\mathcal{F}$ be a class of strict linear orders such that (W, R) is isomorphic to (W, R^{-1}) and $(W, R)^a$ is isomorphic to $(W, R)^b$ for any $(W, R) \in \mathcal{F}$, $a, b \in W$. If $\mathcal{F}$ contains an axis-frame, then the following holds:*

(i) $\mathbf{L}_{\langle \overline{B} \vee \overline{E} \rangle}(\mathcal{F}) = \mathbf{L}(\mathcal{F}) \curlyvee \mathbf{L}(\mathcal{F})$*;*

(ii) $\mathbf{L}_{\langle \overline{B} \vee \overline{E} \rangle}(\mathcal{F})$ *is undecidable;*

(iii) $\mathbf{L}_{\langle \overline{B} \vee \overline{E} \rangle}(\mathcal{F})$ *lacks the finite model property.*

Proof. For a frame $\mathsf{F} = (W, R)$, we have

$$\mathbf{L}(\mathsf{F}) \curlyvee \mathbf{L}(\mathsf{F}) = \bigcap_{a,b \in W} \mathbf{L}(\mathsf{F}^a \curlyvee \mathsf{F}^b) = \bigcap_{aR=b} \mathbf{L}(\mathsf{G}^a \curlyvee \mathsf{F}^b).$$

Due to (7), $\mathbf{L}(\mathsf{F}) \curlyvee \mathbf{L}(\mathsf{F}) = \mathbf{L}_{\langle \overline{B} \vee \overline{E} \rangle}(\mathsf{F})$, that proves (i). Now (ii) and (iii) follow from Theorems 5.9, 5.14. □

Corollary 6.3 *The logics* $\mathbf{L}_{\langle \overline{B} \vee \overline{E} \rangle}(\mathbb{R}, <)$ *and* $\mathbf{L}_{\langle \overline{B} \vee \overline{E} \rangle}(\mathbb{Q}, <)$ *are undecidable and lack the finite model property.*

7 Further results and open questions

The main results of the paper are stated in Theorems 5.9 and 5.14. At the same time, the method of proof is presented in Lemmas 5.6, 5.7 and 5.13. Basing on this method, many other results on products can be transferred to the unimodal case. In particular, various not recursively enumerable $\curlyvee$-products and fragments of HS can be constructed using Theorems 3 and 4 from [7] and Lemma 5.7.

There are many questions about logical properties of $\curlyvee$-products. Let us formulate some of them.

As it was shown in Section 5, some special axioms appear from $\curlyvee$-products of transitive locally-one component frames. However, no complete axiomatizations for logics of this kind are known.

The logic K4 $\curlyvee$ K4 is of special interest. We know that $\Diamond^3 p \to \Diamond^2 p \in \mathrm{K4} \curlyvee \mathrm{K4}$. Does it have the finite model property? Is it decidable? Is it equal to the logic K+$\Diamond^3 p \to \Diamond^2 p$? Note that the finite model property (and, apparently, the decidability) of the latter logic is a long-standing open problem.

Another question was asked by one of anonymous referees: to give an example of decidable logic $\mathrm{L}_1 \curlyvee \mathrm{L}_2$ with undecidable $\mathrm{L}_1 \times \mathrm{L}_2$. Theorem 4.4 now gives the answer in the non-transitive case: if L_1 is an undecidable logic, $\mathrm{L}_2 = \mathrm{T}$, then $\mathrm{L}_1 \times \mathrm{L}_2$ is undecidable, and $\mathrm{L}_1 \curlyvee \mathrm{L}_2 = \mathrm{T}$. At the same time, the author was unsuccessful in finding such an example for transitive L_1 and L_2.

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