Noname manuscript No.
(will be inserted by the editor)

Contemporary modal logic: between mathematics and computer science

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Abstract Modal logic appeared in ancient times as a form of reasoning about necessity and possibility. Contemporary modal logic became one of the tools for solving problems in computer science — both in theory and applications. This was a rather unexpected transition from study of abstract philosophical categories to an actual and practically significant field of modern science. It was prepared by the previous period when modal logic (as well as other parts of logic) extensively developed mathematical methods — algebraic, topological, model-theoretic.

In this article we briefly explain some basic mathematical notions and ideas from modal logic without getting into complicated technical details.

1 Introduction

Logic is usually understood as a certain scientific discipline, a traditional part of philosophy. However, specialists also use the term 'logic' in a narrow sense, as a name of a specific mathematical object. So there exist many particular logics — for example, Classical First-order Logic, Propositional Dynamic Logic (PDL), Girard's Linear Logic, etc. They all can be defined in precise mathematical terms.

Logic in a broad sense is an area of human activity; so it is subject to permanent development and crucial changes. Therefore it cannot be defined exactly. In ancient times logic was aimed to explain how to reason and argue correctly, in the Middle Ages logical reasoning was included in theological arguments, after Renaissance — in the philosophy of science. Mathematical Logic in the

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early twentieth century was almost identical to Foundations of Mathematics ('Metamathematics', in Hilbert's terminology). Nowadays metamathematics is regarded as an important, but not the only part of mathematical logic.

It is difficult to define the subject of mathematical logic also because of variety of smaller fields, with different tasks, methods and styles. So instead of searching for a plausible definition, we just indicate the main components of this area.

At some point the major mathematical results in the field of logic were collected in Handbook in Mathematical Logic [Bar77]. The table of contents of these volumes can give impression of the state of our discipline at that time. The volumes are entitled "Model theory", "Set theory", "Recursion theory", "Proof theory and constructive mathematics".

These directions are still existing and developing; at the same time, the past forty years witnessed the incredible computer revolution, and the growth of theoretical computer science. So recursion theory became just a part of this huge area. Proof theory (as well as constructive mathematics) is now also influenced by the computer science paradigm. Partly by the same reason study of different nonclassical logics has moved from periphery to the centre of mathematical logic. Note that among the great variety of nonclassical logics only some constructive logics and theories (especially, intuitionistic) and the basic modal logic of provability were chosen for Barwise's Handbook.

Typical problems investigated by mathematical logicians are: consistency of axiomatic theories; algorithmic decision problem for different deductive systems; semantic completeness of axiomatic theories in different semantics; definability of properties of mathematical structures in different languages; special syntactic properties of theories — such as interpolation property or disjunction property. In many cases these investigations involve methods from various "non-logical" fields of modern mathematics.

Logics in the narrow sense are formal mathematical objects (certain sets or structures), and they can be precisely defined. So Mathematical Logic as science studies logics as objects. A similar double terminology occurs in other areas of mathematics — for example, Algebra studies algebras, Geometry studies geometries, Topology studies topologies etc. Modal Logic is not exceptional — it studies modal logics.

In this paper we give a brief introduction to modal logic. We begin with general problematics of mathematical logic (section 2), then formulate main definitions and some results from theory of modal logics (sections 3, 4); this material is illustrated by a certain example — the modal logic of inequality (section 5).

2 Some basic notions

Primary notions of logic are 'word' (*logos*) and 'sentence' (*sententia* — meaning in Latin not only a particular syntactic construction, but a specific argument).

At all times logic dealt with words (or texts) written in certain languages. Thus logic is closely connected to other disciplines studying words and languages (such as linguistics or combinatorial group theory). However, logicians are interested both in words and their meanings, so the languages studied in logic usually have syntactic and semantical components, just as any natural language.

In mathematical logic this dualism manifests itself in cooperation between two areas: proof theory and model theory. Every working logician deals with both of them — proofs justify our models, while models verify our proofs. Here one can find an analogy to correlation between theories and experiments in physics and other natural sciences. However, models in pure mathematical logic are still abstract and theoretical, while applied logic can use models simulating reality in the same way as in applied mathematics or physics.

2.1 Syntax of formal languages and calculi

In the historical perspective of the last two centuries the development of mathematical logic was rapid, and the changes involved formal logical syntax as well. The original idea of presenting formal logic as the Boolean logic of propositions transformed into rather ambitious projects of axiomatizing the whole mathematics on the base of classical first-order logic, Hilbert's formalism and Bourbakism. These programs were not realized after all, but they emphasized an important role of classical first-order language as a candidate for the universal language of science.[1] (The Leibnitzean dream of creating such a language still exists.) Modern logic deals with a great variety of special formal languages. They include abstract languages — propositional, first-order, high-order, infinitary. Their further specifications are languages for applied logic, such as programming languages, query languages for databases etc. Diversity of logical languages generates multiple options for their expressive strength.

Logical syntax usually includes two levels. The first level describes *correct language expressions* (especially, propositions), and the second level describes *correct proofs* allowing us to deduce new propositions (theorems) from a fixed set of postulates (axioms). Both descriptions can be done in the same formal style.

A formal definition of syntax begins with a choice of an *alphabet* — a certain set of *symbols* (or *letters*); sequences of letters are called *words*.[2] Some of the words are declared *well-formed expressions*, and they constitute what is called a *formal language*. In their turn, some expressions represent propositions; they are called *formulas*. Also there may be formal expressions for other objects of discourse, such as 'sets', 'numbers', 'programs', 'agents', 'states' etc.

[1] Let us also mention a recent project by V. Voevodsky "The Univalent Foundations Program" proposing to replace the first-order predicate logic with intuitionistic type theory [The13].

[2] Words are usually finite, but mathematical logicians also consider infinitary languages allowing for infinite words.

A formal language is usually described as a system of rules (a calculus) generating complex expressions from simpler ones. For example, this system can be presented as a context-free formal grammar (or in Backus normal form). A procedure generating a certain expression is a *derivation* in this formal grammar.

Logical calculi are at the second level of logical syntax. Such a calculus consists of a set of formulas called *axioms* and a set of *inference rules*. A formal *proof* (or *derivation*) is then arranged as a process generating new formulas (*theorems*); every new theorem is obtained by applying inference rules to axioms and/or earlier theorems. This is exactly the same as a correct argument in traditional logic — a sequence of sentences obtained from basic facts by using rules.[3].

An important feature of a logical calculus is *efficiency*. This means ability (or more exactly, existence of an algorithm) for checking proof correctness. Efficiency puts restrictions on sets of axioms and rules and on definitions of formulas, because it is desirable that formulas and their finite sequences (proofs) could be inputs of programs for proof checking.

A well-known standard example is *classical predicate logic* [Kle02]. Here the first level of syntax contains two types of well-formed expressions — *terms* and *formulas*. Complex expressions are constructed from atomic ones by recursion.

Atomic terms are *individual constants and variables*, and complex terms are built from them by applying *function symbols* (so for example, if x, y are variables, 1 is a constant and $+, \times$ are function symbols for addition and multiplication, then $(x + 1) \times y$ is a complex term).

Atomic formulas are constructed from terms and *predicate symbols*; for example, $(x + 1) \times y < x + x$ is an atomic formula (if $<$ is a predicate symbol and $x, y, 1, +, \times$ are the same as above). Complex formulas are obtained by applying *logical connectives* and *quantifiers* to atomic formulas. For example, $\forall x \exists y\, ((x + 1) \times y < (x + x) + 1)$ is a formula.

At the second level there is an axiomatic system of predicate logic. It can be presented in different equivalent versions — Hilbert-type systems, sequent systems, natural deduction systems. The reasons for this diversity are both theoretical and practical — sequent systems are convenient for proof analysis, Hilbert systems have simpler formulations, while natural deduction is better for correlation with the usual mathematical proofs.

Finally we remark that syntactic study of logical languages can be arranged within the general context of mathematical linguistics. However, there is a difference between logic and linguistics in the goal of this analysis. Logicians are interested mainly in *what* can be expressed in a certain language, while linguists answer the question: *how* a certain language expresses things?[4]

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[3] Aristotelean syllogistic is a less formal example of a logical calculus; however, it can be presented in a formal way as well [Łuk57].

[4] Of course there is no clear border here. Let us illustrate this by an example. Typical results in mathematical linguistics are the theorems by Gaifman [BHGS60] and Pentus [Pen95]: a language is generated by a context-free grammar if and only if it is generated by a Lambek grammar. From the viewpoint of linguistics, this shows that description of natural

2.2 Semantics of formal languages and calculi

Semantical analysis appears in several sciences (semiotics, logic, linguistics, computer science, psychology) in different versions. From the general semiotic viewpoint semantics associates *meanings* with language expressions. However, in logical semantics we need not only to understand what a certain proposition *means*, but also to know if it is *true* or not. So propositions should be associated with *truth values*.

Generally speaking, in mathematical logic, to define a semantics means to define the notion of a *model*. However, the term 'model' is used in different senses: we distinguish models of languages, models of formulas, and models of theories.[5]

To construct a semantics for a logical language L we associate L with a certain class of mathematical structures, the *models of L*. Well-formed expressions of L are then interpreted in these models. In particular, we regard formulas of L as propositions (statements) about models of L. Of course, the truth value of a formula depends on the chosen model. For example, the first-order formula $\forall x \exists y\, (y < x)$ is true on the set of integers with the standard ordering, but false on the set of positive integers.

More exactly, for any model M of L and for any formula φ in L we should formally define when φ is true in M. Usually such a definition is given by induction on the length of φ.

A *model of a formula* φ in a language L is a model of L, where φ becomes true.

In the same way we can talk about models of axiomatic calculi (axiomatic calculi are often called *axiomatic theories*). Namely, M is called a *model of a theory C* if all theorems of C are true in M.

Study of theories and models goes in two directions.

On the one hand, we can be interested in models of particular theories, such as models of Peano Arithmetic, models of the theory of groups (i.e., groups) etc. In general, mathematicians often define specific classes of structures by axiomatizing their properties — the well-known examples are the axioms of topological spaces or Kolmogorov's axioms of probability theory. These definitions can also be arranged as formal theories and studied by logical methods. Axiomatic definitions are common in mathematics of the last century; in particular, they are systematically used in Bourbaki's treatise "Elements of Mathematics". However, the logical analysis of the corresponding theories has not yet been done systematically.

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language syntax using generating grammars in Chomsky style is equivalent to description through grammar categories in Ajdukiewicz style. But from the viewpoint of mathematical logic, these results show that two certain types of axiomatic calculi are equivalent (i.e., they can prove the same theorems).

[5] Formal semantics developed in mathematical logic essentially influenced other disciplines, linguistics and computer science. E.g. language analysis by Montague grammars or semantic web approach to data analysis include truth values and models.

On the other hand, given a model M of a language L, one can consider the set $T(M)$ of all formulas of L that are true in M.[6] $T(M)$ is called the *theory of* M (in this language). Different languages can be used for the same structure producing a variety of theories. Theories of this kind are intensively studied by mathematical logicians.

2.3 Soundness and completeness

Correlation between syntax and semantics of a logical calculus manifests itself in properties of soundness and completeness.

If the formulas chosen as axioms for our calculus C are true in a structure M and the truth in M is preserved by the inference rules of C, then C is called *sound with respect to* M. Obviously, in this case all the theorems of C become true in M, so M is the model of C. Thus we have $[C] \subseteq T(M)$, where $[C]$ denotes the set of theorems of C.

A stronger property is *completeness*: a calculus L is called complete with respect to a model M if the set of its theorems coincides with $T(M)$: $[C] = T(M)$.[7]

Completeness is one of the crucial properties in mathematical logic, but it may be quite difficult to achieve it. The famous Gödel Incompleteness Theorem implies that Peano Arithmetic (PA) is incomplete w.r.t to its standard model, the set of natural numbers $\mathbf{N}$. Moreover, there is no hope to make it complete by adding finitely many missing axioms, because in principle the first-order theory $T(\mathbf{N})$ cannot be efficiently axiomatized.[8] This obstacle appears for many other theories, where PA can be interpreted.

However, in the 20th century complete first-order theories were thoroughly investigated by model theorists. In many cases they were successfully axiomatized. Typical examples (both found by Tarski) are the first-order theory of the field of the reals $T(\mathbf{R})$ and the first-order precise axiomatization of Euclidean geometry of the real plane.

Completeness is a very desirable property, and let us mention one of its applications. If a calculus C is complete w.r.t M, we can forget about M and study only theorems of C. Also, C can be complete of w.r.t. another, perhaps a simpler model M'. In this case we can replace M by M'; in fact,

$$T(M) = [C] = T(M'),$$

i.e., M and M' verify the same sentences. Then the models M and M' are called *equivalent* (in the language L); *elementary equivalent* if L is a first-order language. If we are interested in specific properties of M rather than proofs in C, we can construct an appropriate M' and forget C after that. Such

[6] For technical reasons, in the case of first-order languages $T(M)$ is usually defined as the set of sentences — formulas without free variables.

[7] Some authors call C complete if $T(M) \subseteq [C]$. In this terminology the equality $[C] = T(M)$ means that L is both sound and complete.

[8] This notion will be discussed later on.

an approach was used by A. Robinson in his nonstandard analysis: to prove theorems from the real analysis, i.e., properties of the field of the reals $\mathbf{R}$, one can construct an elementary equivalent model, the field of hyperreals $^*\mathbf{R}$ (containing infinitesimals) and prove the required properties for this model. Cf. [Gol12] for the details.

2.4 Enumerability, decidability, computational complexity

Yet at the beginning of the last century mathematicians had a hope that all precisely formulated mathematical problems could be solved efficiently. However, this hope was destroyed afterwards; now we know plenty of undecidable algorithmic problems in different parts of mathematics.

Every algorithmic problem can be put as a computational task for a function transforming words into words (maybe, in another language). In particular, it may be a *decision problem*: to determine whether a given word in a language X belongs to its certain sublanguage Y; more exactly, this is a decision problem for Y w.r.t. X. Usually in this case the larger language X has an explicit description, i.e., the decision problem for X w.r.t. the set of all words (in the given alphabet) can be solved. To solve a decision problem means to construct an algorithm, which for every input a from X gives an answer 'yes' if $a \in Y$ and 'no' otherwise. If such an algorithm exists, the set Y is called *decidable.*

A related computational task is *generation* of a set Y. This means constructing an algorithm that produces elements of Y one by one (perhaps, with repetitions).[9] If such an algorithm exists, the language Y is called *(computably) enumerable.*[10]

Again, a larger set X is usually enumerable by a standard procedure. Then we have

Fact 1 If Y is decidable in X, then Y is enumerable.

Indeed, to generate a nonempty Y just subsequently generate the words of X and for each of them decide whether they are in Y.

Commonly used logical languages are generated by inductive definitions equivalent to context-free grammars. This always gives us a computable enumeration, together with a decision procedure — by standard methods from mathematical linguistics. So the first level of logical syntax is rather simple from the computational viewpoint.

But situation at the second level of syntax is far from nice. In fact, even if we give a very good algorithmic definition of a logical calculus and use an exact notion of proof, finding particular proofs can make a serious problem.

Let us formulate this in more precise terms. First note the following

[9] More exactly, given an input number n, this algorithm should produce the n-th element in an enumeration of Y.

[10] The empty set is also enumerable, by definition.

Fact 2 If C is an axiomatic theory with a decidable set of axioms and a decidable set of rules, then

(1) the set of proofs in C is decidable,
(2) the set $[C]$ of theorems of C is enumerable.

Remarks on the proof. (1) In fact, given a sequence of formulas, for every its member one can successively check if it is an axiom or if it is obtained from earlier formulas by one of the rules. So in principle every argument in a precisely formulated theory can be checked automatically — in this respect the dreams of philosophers and logicians of past times are realizable.

(2) Enumeration of the set $[C]$ can be constructed as follows. Decidability of the set of proofs implies its enumerability (fact 1). So we can start a procedure generating proofs and subsequently extracting formulas occurring in them — actually, for every proof we can take just its last formula.

In a more general setting, a set of words Y is called *efficiently axiomatizable* if it coincides with the set of theorems of some calculus with a decidable set of axioms and inference rules. So by the above argument we have

Fact 3 Every efficiently axiomatizable set of words is enumerable.

Moreover, in many cases for logical calculi efficient axiomatizability and enumerability are equivalent (Craig's theorem). For example, this is true for first-order theories and modal logics considered below; however, sometimes this equivalence does not hold [KLR17].

Next, a logical calculus is called *decidable* if the set of its theorems is decidable. So for a decidable calculus C there must be a *decision procedure* — an algorithm solving the decision problem. For every input formula φ, it gives the answer 'yes' if φ is a theorem of C and 'no' otherwise. From the practical side, we may ask for more: if the answer is 'yes', it is desirable to get a proof of φ. If C is decidable, this is also possible: start a computable enumeration of all proofs and check if φ occurs in them. Given that φ is a theorem, this procedure will stop at some stage. Of course, such an algorithm is far from optimal; in practice there exist faster methods of automated proof search.

By fact 1, every decidable calculus is enumerable. The converse is not true:

Fact 4 There exist enumerable, but undecidable calculi.

The question about decidability of a certain calculus may be very nontrivial. After the first undecidability results were obtained in the 1930s (by A. Church — for Peano Arithmetic and Classical predicate calculus), investigation of decidability has become a large area of mathematical logic; an overview can be found in [Bar77], v.3.

Undecidability phenomena lead us to a general problem: what kind of formal theories should be developed? On the one hand, we would like to have a language formalizing various properties of mathematical structures and prove theorems in this language. On the other hand, if we make our formal theories too powerful, they become undecidable. In many cases, classical first-order theories of particular structures $T(M)$ are undecidable (for example, if M is the ring of integers $\mathbf{Z}$ or the field of rationals $\mathbf{Q}$, cf. [TMRR10]). It is easy to see that if $T(M)$ is undecidable, then it is also non-enumerable, so there is no hope to axiomatize it efficiently in these cases.

Another important aspect is the computational cost for decidable theories: how much of resources (time or memory) can decision procedures take? These questions are studied by *computational complexity theory*, also a large area of logic and computer science. This area was intensively developing in recent decades, but many fundamental problems remain open here. The most famous of them is the problem of equality of complexity classes P and NP.

Therefore, in mathematical logic we have to compromise between the expressive power and algorithmic properties of formal theories. Full harmony is impossible; a partial compromise is sometimes achieved in modal logics.

3 Modal logic

Evolution of this area between philosophy, mathematics, and computer science is described in a large paper by Goldblatt [Gol03] in many details. Let us briefly point out some stages of this development.

Modal logic as a rather modest part of logic emerged in ancient times within philosophy. Many famous philosophers contributed in modal logic studies, beginning from Aristotle (for example, Ockham, Buridan, Leibnitz, Kant, Peirce, and others). Only in the 20th century modal logic acquired its main technical toolkit and became a separate discipline within mathematical logic. Finally, its broad practical applications were found at the end of the last century, thus bringing modal logic to computer science. So modal logic first supposed for analysis of philosophical categories (especially, necessity and possibility), after a period of mathematical formalization, became an instrument for solving specific practical problems.

3.1 From modal syntax to semantics

Probably, the first definition of modal logics as axiomatic calculi (in which a logic is presented as a set of theorems of a certain calculi) was given by a well-known American philosopher Clarence Lewis [LL32].

Some attempts of formalizing modalities were made earlier ([Mac06], [Lew18]), but these works were about the choice of notation and its informal understanding, without any exact definitions of calculi or semantics. The history of this period is discussed in [Gol03].

The most famous calculi introduced by Lewis (**S4** and **S5**) will be discussed later on. First, let us define the language of propositional modal logic.

Similarly to classical (Boolean) formulas, modal formulas are constructed from a countable set of *propositional variables* PV by applying propositional connectives (with parentheses). There are usual binary connectives $\vee$ (*disjunction*), $\wedge$ (*conjunction*), $\rightarrow$ (*implication*), the unary connective $\neg$ (*negation*).

We also have propositional constants (or 0-ary connectives) $\bot$ (*falsity*) and $\top$ (*truth*). There are additional unary modal connectives $\Diamond$ (*diamond*) and $\Box$ (*box*); their traditional reading is ‘possibly’ and ‘necessary’.

Beginning from Lewis's work, numerous calculi axiomatizing “modal laws” were constructed. The first interpretation of “necessity” and “possibility” was intuitive. But intuition does not help in choosing true logical laws for these notions.

For example, it is not clear if the implication $\Diamond\Diamond\varphi \to \Diamond\varphi$ is true. I.e., is some fact φ possible given that it is possible that φ is possible? Perhaps, the answer depends on our understanding of “possibility”. In particular, $\Diamond\Diamond\varphi \to \Diamond\varphi$ should be true if $\Diamond\varphi$ is understood as “φ may happen in the future”; but this implication should be false if $\Diamond\varphi$ is understood as “φ may happen tomorrow”.

So we cannot exactly identify true logical properties of modalities before we formalize an intuitive notion of truth. There are two ways to obtain partial solutions for this problem.

On the one hand, we can postulate a small number of intuitively true laws as axioms of a certain calculus; then theorems of this calculus also become “true”. This option was chosen by Lewis and other his contemporaries.

On the other hand, we can define models of our language (i.e., describe semantics) and find out if a given proposition is valid, i.e. true in every model. If it is not always true, we can find out, in which models it is true. At the same time the adequacy of our formal definition of semantics remains an open question.

By the 1950s several modal semantics were introduced (in different styles, formal or informal). The most famous of them is relational *Kripke semantics* giving an intuitively clear and mathematically precise interpretation of modal language.

3.2 Kripke semantics

Definition 1 *A Kripke frame*, or just a *frame*, is a pair $F = (W, R)$, where W is a non-empty set, R is a binary relation on W. A *valuation* over a frame F is a map $V : PV \longrightarrow \mathcal{P}(W)$ (where $\mathcal{P}(W)$ is the set of all subsets of W), i.e., $V(p) \subseteq W$ for any variable $p \in PV$; the pair $M = (F, V)$ is then called a *Kripke model* over F. The set W is called the *domain* of F (and M), R is the *accessibility relation* in F (and M).

The elements of the domain W are traditionally called *possible worlds* (the notion introduced by Leibnitz), but in modern works they are also called ‘points’, ‘states’, ‘moments of time’ (if temporal modalities are discussed). The main feature distinguishing Kripke semantics from Boolean semantics of classical formulas is that truth values of formulas in a Kripke model depend not only on this model, but also on possible worlds. A formula true at one world may become false at another world — this is quite natural in everyday logic.

The truth of a modal formula in points of a Kripke model $M = (F, V)$ is formally defined by induction on the length of the formula (the notation $M, w \vDash \varphi$ reads as *"formula φ is true at point w of model M"*):

$$
\begin{array}{lcl}
M, w \vDash p & := & w \in V(p); \\
M, w \vDash \neg\varphi & := & M, w \nvDash \varphi; \\
M, w \vDash \varphi \vee \psi & := & M, w \vDash \varphi \text{ or } M, w \vDash \psi; \\
M, w \vDash \varphi \wedge \psi & := & M, w \vDash \varphi \text{ and } M, w \vDash \psi; \\
M, w \vDash \varphi \to \psi & := & M, w \nvDash \varphi \text{ or } M, w \vDash \psi; \\
M, w \vDash \bot & := & \text{false}; \\
M, w \vDash \top & := & \text{true}; \\
M, w \vDash \Diamond\varphi & := & \text{there exists } v \text{ such that } wRv \text{ and } M, v \vDash \varphi; \\
M, w \vDash \Box\varphi & := & M, v \vDash \varphi \text{ for all } v \text{ such that } wRv.
\end{array}
$$

(the sign ':=' is an abbreviation for ≪means by definition that≫).

The meaning of this definition is the following. Boolean connectives at every point behave as in classical logic, and necessity is understood as truth in every accessible point. Kripke refers this viewpoint to Leibnitz who had a similar semantics for necessity (without further formal specification), but with the universal accessibility relation. Thus in Leibnitz's approach every point is accessible from every point; necessity means truth in all worlds, and possibility means truth in some worlds (see later, section 5.4).

Diodorus Cronus (the beginning of the 3rd century B.C.) could be regarded as another ancestor of Kripke. In his Master Argument he used a temporal interpretation of modalities: something is possible if either it is happening now, or will happen at some future time; respectively, something is necessary if it is happening now and will happen always in the future. In Kripke semantics this understanding of modalities corresponds to the case when points are moments of time and the accessibility relation is (non-strict) precedence in time.

Note that a priori we do not put any special restrictions on R. From the mathematical viewpoint, the nature of points of a model does not matter either. We can regard them as "possible worlds" (as in Kripke's approach) or as "moments of time" (as in works in temporal logics) or as "states of a computing system" (as in works on dynamic logics).

The meaningful interpretation of "propositions" p is not involved in this consideration; we are interested only in the value $V(p)$, and we do not care about the meaning of p itself.

A formula φ is said to be *valid* on a frame (W, R) if it is true at any point under any valuation; this is denoted by $(W, R) \vDash \varphi$.

Now note that the truth definition allows us to extend the function V to all formulas by putting $V(\varphi) := \{w \mid M, w \vDash \varphi\}$. In fact, instead of defining truth at points, we can define the values $V(\varphi)$ by induction:

$$\begin{array}{lcl} V(\neg\varphi) & := & -V(\varphi); \\ V(\varphi \vee \psi) & := & V(\varphi) \cup V(\psi); \\ V(\varphi \wedge \psi) & := & V(\varphi) \cap V(\psi); \\ V(\varphi \to \psi) & := & -V(\varphi) \cup V(\psi); \\ V(\bot) & := & \emptyset; \\ V(\top) & := & W; \\ V(\Diamond\varphi) & := & R^{-1}(V(\varphi)); \\ V(\Box\varphi) & := & -R^{-1}(-V(\varphi)). \end{array}$$

Here $\cup$, $\cap$, $-$ are the set-theoretic operations of union, intersection, and complement (to W), $R^{-1}(X)$ is the *inverse image of* X *under* R:

$$R^{-1}(X) = \{u \mid \exists w \in X \ uRw\}.$$

As the definition of $V(\varphi)$ is inductive, the values of the function V can be chosen from not the whole $\mathcal{P}(W)$, but a smaller family $\mathcal{V}$ of subsets of W. However, to make such a restricted definition sound, we need the following conditions on $\mathcal{V}$:

- $X \in \mathcal{V} \Rightarrow -X \in \mathcal{V}$,
- $X, Y \in \mathcal{V} \Rightarrow (X \cup Y) \in \mathcal{V}$,
- $X \in \mathcal{V} \Rightarrow R^{-1}(X) \in \mathcal{V}$.

Thus, the operations of union and complement should preserve membership in $\mathcal{V}$. Of course, then intersections are also preserved, since $X \cap Y = -((-X) \cup (-Y))$, by De Morgan's law. Therefore, to evaluate all our formulas, $\mathcal{V}$ must be a *Boolean algebra.*

Also $\mathcal{V}$ must be closed under taking inverse images w.r.t. R. The set $\mathcal{V}$ with Boolean operations and the above defined function R^{-1} is an example of a *modal algebra.* In modal logic these algebras play the same role as Boolean algebras in classical logic. A general definition of a modal algebra will be given later on.

A Kripke frame (W, R) together with a set $\mathcal{V}$ satisfying these closure conditions is called a *general Kripke frame.* A modal formula φ is said to be *valid* on a general Kripke frame $(W, R, \mathcal{V})$ if $V(\varphi) = W$ for any valuation V taking values in $\mathcal{V}$.

General frames yield a slightly different version of Kripke semantics. It is less visible, but has some technical preferences; they will be discussed in section 3.5.

3.3 An example of a modal calculus

Once we have defined semantics for formulas of a given formal language, we can evaluate their truth. Semantics also allows us to describe properties of mathematical structures in the chosen language.

Consider, for example, the formulas $p \to \Diamond p$ and $\Diamond\Diamond p \to \Diamond p$. What do they say about an accessibility relation? By applying the truth definition, one can see that

$$\begin{array}{lll} (W,R) \vDash p \to \Diamond p & \text{iff} & R \text{ is reflexive;} \\ (W,R) \vDash \Diamond\Diamond p \to \Diamond p & \text{iff} & R \text{ is transitive.} \end{array}$$

Recall that a relation R is *reflexive*, if xRx for all points x, and *transitive*, if xRy and yRz imply xRz.

These and many other examples show that modal formulas can express various properties of the accessibility relation. We point out that in some cases a property expressible by a modal formula cannot be expressed by classical first-order formulas. A well-known example of this kind is Löb's formula

$$\Box(\Box p \to p) \to \Box p$$

stating that R is transitive and Nötherian, i.e., there are no infinite sequences (chains) of the form $x_0 R x_1 R x_2 ...$ (perhaps, with repetitions). On the other hand, some first-order properties (for instance, irreflexivity) cannot be expressed by modal formulas.

Thus, sometimes the class of corresponding frames can be described in classical first-order language (then the modal formula is called *elementary*), but this is not always the case. In general the set of elementary modal formulas is undecidable [Cha91].

Let us now turn from single modal formulas to modal calculi. As an example, we first consider one of Lewis's modal systems traditionally denoted by **S4**.[11]

Definition 2 **S4** is the set of theorems of the calculus given by the following three groups of axioms (where p, q are propositional variables):

(1) all classical tautologies,

(2)

$$\begin{array}{l} \Box(p \to q) \wedge \Box p \ \to \ \Box q, \\ \Diamond p \ \leftrightarrow \ \neg\Box\neg p, ^{12} \end{array}$$

(3)

$$\begin{array}{l} \Diamond\Diamond p \ \to \ \Diamond p, \\ p \ \to \ \Diamond p; \end{array}$$

[11] We provide an equivalent formulation close to the one proposed by Gödel in [Göd33].

[12] As usual, $A \leftrightarrow B$ is an abbreviation for $(A \to B) \wedge (B \to A)$.

and the inference rules of *Modus Ponens* (MP), *Substitution* (SUB), and *Necessitation* (NEC):

$$\frac{\varphi, \qquad \varphi \to \psi}{\psi}\ \text{MP} \qquad\qquad \frac{\varphi(p)}{\varphi(\alpha)}\ \text{SUB} \qquad\qquad \frac{\varphi}{\Box\varphi}\ \text{NEC}$$

In the substitution rule, $\varphi(\alpha)$ is obtained from $\varphi(p)$ by replacing each occurrence of p with α.

Initially **S4** was introduced as a formalization of philosophical notions of necessity and possibility. Later other interpretations and applications of **S4** were found.

In particular, we have the following semantics for this system:

Theorem 1 **S4** *is the set of formulas that are valid on all transitive reflexive frames.*

(For the proofs of this and the next two theorems see, e.g., [CZ97] and [BdRV01].) Thus, to "understand the laws" of **S4** one can study formulas valid on transitive reflexive frames.

In fact, this theorem can be strengthened:

Theorem 2 *A formula φ belongs to* **S4** *iff φ is valid on every transitive reflexive frame (W, R) such that $|W| \leq 2^{|\varphi|}$, where $|W|$ is the cardinality of W, $|\varphi|$ is the length of φ.*

It follows that **S4** is decidable: there exists an algorithm that for every input formula φ gives an answer 'yes' if $\varphi \in \mathbf{S4}$ and 'no' otherwise.[13] Moreover, every non-theorem is falsified in a "relatively small" frame — its size is exponentially bounded in the length of φ. Furthermore, the following holds:

Theorem 3 **S4** *is* PSPACE-*complete*[14].

This theorem was proved in [Lad77]. It turns out that many other natural modal systems are PSPACE-complete [Lad77, Spa93], and some of them are of the same complexity as classical logic (coNP-complete). In fact, study of complexity in modal logic and further interaction between modal logic and computer science began from this Ladner's result.

[13] More generally, if a modal calculus C has only finitely many axioms and is complete with respect to a class of finite frames, then it is decidable (Harrop's Theorem). Indeed, on the one hand, the set of theorems of C is enumerable. On the other hand, we can enumerate all finite frames and for each of them effectively decide whether all theorems of C are valid — we have only to check the validity of a finite set of axioms. It follows that the set of non-theorems of C is enumerable. Now decidability follows from Post's theorem [Mal70].

[14] Recall that a decision problem Y is in the class PSPACE if it is decidable within a polynomial (in the length of input) amount of space. A problem Y is PSPACE-complete if it is in PSPACE and every problem in PSPACE is polynomial time reducible to Y. See, e.g., [AB09].

3.4 Normal modal logics

As we noticed before, two of **S4**-axioms (group (3)) express transitivity and reflexivity of a frame (it other words, they *define* the class of all transitive reflexive frames).

However, in many cases they consider modal logics without these axioms and respectively, frames with arbitrary accessibility relations.

Definition 3 *A normal modal logic* is the set of theorems of a calculus containing the groups of axioms (1), (2) from Definition 2 and the inference rules MP, NEC, and SUB.

In this paper we consider only normal modal logics. The smallest such logic is denoted by **K**. It is axiomatized by the calculus containing the axioms (1) and (2) (and rules as in Definition 3). Other modal logics contain some extra axioms. We denote the modal logic with a set of extra axioms Ψ by $\mathbf{K}+\Psi$. (In particular, the set of all modal formulas is a logic, which is called *inconsistent*.)

Theorem 4 (Soundness theorem for Kripke semantics) *The set of modal formulas valid on a Kripke frame is a normal modal logic.*

The proof readily follows from two observations. First, the axioms of **K** are valid on every frame. Second, the set of modal formulas valid on a Kripke frame is closed under the rules MP, NEC, and SUB (this is trivial for MP and Nec; for SUB, this is an easy exercise).

Corollary 1 *The set of modal formulas valid on a class $\mathcal{F}$ of Kripke frames (i.e., on every frame from $\mathcal{F}$) is a normal modal logic.*

This set is called the *modal logic of* $\mathcal{F}$ and is denoted by $ML(\mathcal{F})$. Logics of this kind are called *complete* (more precisely, *Kripke complete*). We also say that the logic $ML(\mathcal{F})$ is *complete with respect to the class* $\mathcal{F}$.

The following theorem is an analogue of Theorems 1,2, and 3 for the logic **K**; now instead of transitive and reflexive frames, we consider arbitrary frames.

Theorem 5

(1) **K** *is the logic of the class of all frames, moreover, of the class of all finite frames.*
(2) $\varphi \in \mathbf{K}$ *iff* φ *is valid on every frame* (W, R) *such that* $|W| \leq 2^{|\varphi|}$.
(3) **K** *is decidable and moreover,* PSPACE*-complete.*

For the proof see, e.g., [CZ97] or [BdRV01].

As we have mentioned, not every modal logic shares good semantic and algorithmic properties with the logics $\mathbf{K}, \mathbf{S4}$. Not every modal logic is Kripke complete (perhaps, the simplest example of an incomplete logic is the logic $\mathbf{K} + \Box(\Box p \leftrightarrow p) \to \Box p$; see, e.g., [Gol92]). Moreover, there are continuum many incomplete modal logics [Blo80].

At the same time there exists a continuum of complete modal logics [Fin74]. Among them, continuum many are undecidable (or even non-enumerable), because there are only countably many enumerable logics[15].

3.5 Algebraic semantics

Although some modal logics are incomplete in Kripke semantics, there exists another, algebraic semantics for modal formulas, in which all normal modal logics happen to be complete. This semantics emerged in the 1930s, soon after the first modal calculi were defined.

In algebraic semantics the main objects, where modal formulas are evaluated, are so-called modal algebras.

We assume that the reader is familiar with the definition and basic properties of Boolean algebras (see, e.g., [BS81]). We will use the same notation for the Boolean operations as for the logical connectives: $\vee$, $\wedge$, $\neg$, $\bot$, $\top$. A *modal algebra* is a Boolean algebra with extra unary operations $\Box$ and $\Diamond$ satisfying the identities

$$\Diamond(a \vee b) = \Diamond a \vee \Diamond b;$$
$$\Diamond\bot = \bot;$$
$$\Box a = \neg\Diamond\neg a.$$

.

For example, a topological space induces a modal algebra — the Boolean algebra of its subsets with extra operations of *closure* and *interior*. The operation $\Diamond$ sends a set Y to its closure (the set of all adherent points of Y). The interior operation $\Box$ sends a set Y to its interior (the set of all interior points of Y).

A modal algebra can also be obtained from a Kripke frame (W, R): take the Boolean algebra of subsets of W with the preimage operation $\Diamond$ mapping $Y \subseteq W$ to $R^{-1}(Y)$, and put $\Box Y = \neg R^{-1}(\neg Y)$. Also, if $(W, R, \mathcal{V})$ is a general Kripke frame (see section 3.2), then $\mathcal{V}$ constitutes a modal algebra, *the modal algebra of* $(W, R, \mathcal{V})$.

Modal formulas can be regarded as terms in the signature of modal algebras. So we can define the notion of truth of a modal formula in a modal algebra:

Definition 4 A modal formula φ *is true* in a modal algebra A, if the identity $\varphi = \top$ holds in A.

Theorem 6 *The set of modal formulas that are true in a modal algebra A is a normal modal logic.*

This set is called the *modal logic of A*.

In algebraic semantics all modal logics are complete:

[15] Every enumerating program can be saved in a file; the content of a file is encoded with a natural number.

Theorem 7 *For every modal logic L there exists an algebra $Lind(L)$, whose logic is L.*

$Lind(L)$, the *Lindenbaum-Tarski algebra of* L, is constructed in a standard way from modal formulas — as the quotient modulo the equivalence $L \vdash \varphi \leftrightarrow \psi$.

A *variety* is a class of algebras, where a given set of identities is true. Due to algebraic completeness, there is a one-to-one correspondence between normal modal logics and varieties of modal algebras. So study of normal modal logics can be regarded as study of these varieties, and we can use methods of universal algebra in the field of modal logic.

A systematic study of modal algebras was started in the 1940-50s in the works by A. Tarski, J. McKinsey, B. Jónsson and others. In the paper [JT51], Jónsson and Tarski proved a representation theorem for modal algebras, a generalization of the famous Stone's representation theorem for Boolean algebras. In particular, it follows that every modal algebra is isomorphic to the modal algebra of a general frame, and every modal logic is the set of formulas valid on a general frame.

Numerous properties of modal logics can be reformulated algebraically. For example, consider the following property of a logic L: given a finite set of propositional variables, there are only finitely many pairwise non-equivalent in L formulas in these variables. This property is called *local tabularity.* Every locally tabular logic is complete w.r.t. finite frames, so finitely axiomatizable locally tabular logics are decidable. A well-known example of a locally tabular logic is classical propositional logic. Modal logics **K**, **S4** are not locally tabular, while **S5** and **DL** (see below) are. In algebraic terms, local tabularity of a logic means *local finiteness* of its variety: every finitely generated algebra in the variety is finite. Study of local tabularity in modal logic remains a field of great interest. For recent developments cf. [Bez01], [She16], [SS16].

3.6 Other semantics

Besides Kripke and algebraic semantics, two other semantics are important for propositional modal logics — topological (or neighbourhood) and provability. Let us (very briefly) discuss them.

1. In section 3.5 we mentioned that every topological space induces a modal algebra of its subsets. All these algebras satisfy the axioms of **S4**. A detailed study of modal algebras arising in topology was initiated by J. McKinsey and A. Tarski [MT44]. This lead to *topological semantics* of modal logics containing **S4**. *Neighbourhood semantics* proposed by R. Montague and D. Scott is a generalization of topological semantics suitable for a larger class of modal logics — it is sound for every normal modal logic (and even for some weaker systems). In neighbourhood models necessity is understood as truth in some neighbourhood of a given point. See, e.g., [Pac17], [Che80] for details.

Interaction between modal logic and topology is quite active nowadays. It is included in a more general context of spatial logic and spatial reasoning, where modal logic found different applications, cf. [APHvB07].

2. The idea of using modal operators for axiomatization of provability and consistency was put forward by K. Gödel in the 1930s [Göd33]. Realization of this idea was started much later, by Solovay's work [Sol76]. That paper provided a complete modal axiomatization of arithmetical provability, Gödel-Löb logic **GL** (defined as $\mathbf{K}+\Box(\Box p \to p) \to \Box p)$. Since that time modal logic has become an important tool in proof theory.

4 Standard translation

As pointed out in section 2.2, semantics of a formal language proposes a meaningful understanding of its expressions. In modern linguistics, semantics is described as explication of texts, i.e., translating them into another language (of "senses"), cf. [MCP95]. Semantics of modal logic can be represented in the same way — as translation into the language of classical logic.

So Kripke semantics suggests for *standard translation* of modal formulas to classical first-order formulas. It was introduced by J. Van Benthem in early 1970s (and earlier by G.E. Mints for intuitionistic formulas), see [vB88].

Let us recall the definition. Consider classical first-order formulas built from a countable set $\{P_1, P_2, \dots\}$ of unary predicate letters and a single binary letter R. Let $\mathcal{L}_1$ denote the set of all these formulas, and $\mathcal{L}_0$ the set of formulas in $\mathcal{L}_1$ that do not contain $P_1, P_2, \dots$. Also, consider propositional modal formulas built from propositional variables $p_1, p_2, \dots$ By induction for every modal formula φ we define its *standard translation* $\varphi^\bigstar(t) \in \mathcal{L}_1$, a classical first-order formula with a single parameter t, according to the following rules:

$$
\begin{aligned}
&p_i^\bigstar(t) := P_i(t),\\
&(\neg\varphi)^\bigstar(t) := \neg\varphi^\bigstar(t),\\
&(\varphi \to \psi)^\bigstar(t) := \varphi^\bigstar(t) \to \psi^\bigstar(t),\\
&(\varphi \wedge \psi)^\bigstar(t) := \varphi^\bigstar(t) \wedge \psi^\bigstar(t),\\
&(\varphi \vee \psi)^\bigstar(t) := \varphi^\bigstar(t) \vee \psi^\bigstar(t),\\
&(\bot)^\bigstar(t) := \bot,\\
&(\top)^\bigstar(t) := \top,\\
&(\Box\varphi)^\bigstar(t) := \forall x\,(R(t,x) \to \varphi^\bigstar(x)),\\
&(\Diamond\varphi)^\bigstar(t) := \exists x\,(R(t,x) \wedge \varphi^\bigstar(x)),
\end{aligned}
$$

where $\varphi^\bigstar(x)$ is the result of replacing of t by x in $\varphi^\bigstar(t)$ (and renaming bound variables if necessary).

Every Kripke model $M = (F, V)$ on a frame $F = (W, R)$[16] can be regarded as a classical model $M^\star = (W, R, V(p_1), V(p_2), \dots)$ of $\mathcal{L}_1$.[17]

The next simple lemma describes the logical connection between M and $M^\star$.

Lemma 1

(1) Let $M = (F, V)$ be a Kripke model on a frame $F = (W, R)$. Then for every $a \in W$ and modal formula φ,

$$M, a \vDash \varphi \;\Leftrightarrow\; M^\star \vDash \varphi^\star(a).$$

(2) For every frame F and modal formula φ

$$F \vDash \varphi \;\Leftrightarrow\; \textit{for any } V,\ (F, V)^\star \vDash \forall t\, \varphi^\star(t).$$

The proof of (1) is by induction on the length of φ; (2) easily follows from (1).

In follows that in many cases, propositional modal logics can be considered as fragments of classical first-order theories.

Namely, let Σ be a set of sentences in $\mathcal{L}_0$, $Mod(\Sigma)$ the class of all models of Σ (in the classical sense). Clearly, these classical models can be considered as Kripke frames; so we obtain the modal logic $ML(Mod(\Sigma))$.

On the other hand, Σ is a subset of $\mathcal{L}_1$, so it has models in the signature $\{R, P_1, P_2, \dots\}$. These models are of form $(F, V)^\star$, where $F \in Mod(\Sigma)$.

By classical Gödel's completeness theorem we obtain:

Theorem 8 *$\varphi \in ML(Mod(\Sigma))$ iff $\Sigma \vdash \forall t\, \varphi^\star(t)$ (in classical first-order predicate calculus).*

In particular, it follows that $ML(Mod(\Sigma))$ is embeddable in first-order predicate calculus whenever Σ is finite:

Corollary 2 *If Σ is finite, then*

$$\varphi \in ML(Mod(\Sigma)) \quad \Leftrightarrow \quad \vdash \bigwedge \Sigma \to \forall t\, \varphi^\star(t).$$

Proof. By Theorem 8 and Deduction theorem. □

Corollary 3 *If Σ is finite (or even infinite, but decidable),*[18] *then $ML(Mod(\Sigma))$ is enumerable.*

Proof. To obtain a computable enumeration of this modal logic, we generate theorems of the classical theory with the set of axioms Σ. Note that this set is enumerable (Fact 2, section 2.4). Along with this enumeration we can select formulas of the form $\forall t\, \varphi^\star(t)$, since the set of these formulas is clearly decidable. Finally, a modal formula φ can be restored from $\forall t\, \varphi^\star(t)$. □

[16] Strictly speaking, we should not use the same symbol R for a binary letter and a binary relation. However, to simplify the notation, we ignore this formality.

[17] In more details, this means that R (as a symbol) is interpreted by R (as a binary relation), and each P_i by membership in $V(p_i)$.

[18] In fact, the claim also holds for enumerable Σ.

Here a question arises: if modal logics are embeddable in classical first-order theories, why should we study them *per se*? For, classical first-order logic is well developed and its properties have been known for a long time.

However, the situation is not that simple. First, not every modal logic is complete with respect to models of the form $Mod(\Sigma)$ — a well-known counterexample is given by modal provability logic **GL** (see Sections 3.6, 3.3). So for these "irregular" logics classical methods are inapplicable. Second, even for "good" modal logics new methods had to be developed, as classical logic deals with another language and other models. Speaking very informally, classical logic can sometimes understand modal logic, but cannot always help it. Modal logic is another world.

5 The logic of inequality

As another example of a modal logic we consider the *logic of inequality*, or the *Difference logic*. Completeness, decidability, and complexity results for this logic have been known for quite some time ([Seg76], [Seg80], [dSvEB90]). To illustrate basic modal logic tools, we shall give proofs for these results.

For a set X, let $\neq_X$ be the inequality relation on X; formally $\neq_X$ is the set of pairs $\{(x, y) \mid x, y \in X,\ x \neq y\}$.

The *logic of inequality* is defined semantically — as the logic of a class of Kripke frames:

$$\mathbf{L}_{\neq} := ML\{(X, \neq_X) \mid X \text{ is a non-empty set}\}.$$

5.1 Complete axiomatization

Set

$$\mathbf{DL} := \mathbf{K} + \{p \to \Box\Diamond p,\ \Diamond\Diamond p \to \Diamond p \vee p\},$$

and let us show that $\mathbf{L}_{\neq} = \mathbf{DL}$.

First let us characterize the **DL**-frames, i.e., frames validating **DL** (by Soundness theorem 4, this means validity of two additional axioms).

Proposition 1 *For a frame $F = (W, R)$,*

(1) $F \vDash p \to \Box\Diamond p$ *iff* R *is symmetric,*
(2) $F \vDash \Diamond\Diamond p \to \Diamond p \vee p$ *iff* $\forall x \forall y \forall z\,(xRyRz \Rightarrow xRz \vee x = z)$. *In the latter case R is called* weakly transitive.

Proposition 2 *For a non-empty X, $(X, \neq_X)$ is a* **DL***-frame.*

The proofs of both propositions are straightforward, so we skip them.

To prove completeness of **DL** in Kripke semantics we use the *canonical model*. Let us recall its definition.

A set of formulas Γ is said to be *inconsistent* (in a logic L), if $\neg(\varphi_1 \wedge \ldots \wedge \varphi_k) \in L$ for some $\varphi_1, \ldots, \varphi_k \in \Gamma$; otherwise, Γ is *consistent* (again, in L). Γ

is L-*maximal* if it is consistent and all its proper extensions are inconsistent in L.

For a set of formulas Γ put $\Diamond\Gamma := \{\Diamond\varphi \mid \varphi \in \Gamma\}$.

The *canonical frame* of a logic L is $F_L = (W_L, R_L)$, where W_L is the set of all L-maximal sets and R_L is defined as follows:

$$\Gamma\, R_L\, \Gamma' \;:=\; \Diamond\Gamma' \subseteq \Gamma.$$

The *canonical model* $M_L := (F_L, \theta_L)$ of L is its canonical frame with the valuation θ_L:

$$\theta_L(p) := \{\Gamma \in W_L \mid p \in \Gamma\}$$

for each propositional variable p.

The next theorem is one of the most powerful tools in modal logic.

Theorem 9 (Canonical model theorem, CMT)

(1) For all $\Gamma \in W_L$,

$$\varphi \in \Gamma \text{ iff } M_L, \Gamma \vDash \varphi.$$

(2)

$$\varphi \in L \text{ iff } M_L, \Gamma \vDash \varphi \text{ for all } \Gamma \in W_L.$$

A proof can be found, e.g., in [CZ97].

A modal logic is called *canonical* if it is valid on its canonical frame.

Lemma 2 *Every canonical logic is complete.*

Proof. By CMT, L contains the logic of its canonical frame. On the other hand, L is contained in the logic of F_L, since $F_L \vDash L$ by canonicity. Thus we have Kripke-completeness and moreover, completeness with respect to F_L. □

Now the following argument proves completeness for **DL**.

Proposition 3 *The canonical frame $F_{\mathbf{DL}}$ is symmetric and weakly transitive.*

Proof. We abbreviate $R_{\mathbf{DL}}$ to R.

To prove symmetry, assume that $\Gamma R \Delta$ for some maximal sets Γ, Δ, and show that $\Delta R \Gamma$. Let $\varphi \in \Gamma$. Notice that the formula $\varphi \to \Box\Diamond\varphi$ is in **DL**. By CMT we have first, $M_{\mathbf{DL}}, \Gamma \vDash \varphi$, and second, $M_{\mathbf{DL}}, \Gamma \vDash \varphi \to \Box\Diamond\varphi$. Thus $M_{\mathbf{DL}}, \Gamma \vDash \Box\Diamond\varphi$. Since $\Gamma R \Delta$, it follows that $M_{\mathbf{DL}}, \Delta \vDash \Diamond\varphi$, and so $\Diamond\varphi \in \Delta$ (again by CMT). Thus $\varphi \in \Gamma$ implies $\Diamond\varphi \in \Delta$. By definition, $\Delta R \Gamma$.

Let us check the weak transitivity.

Suppose $\Gamma R \Delta R \Sigma$ and $\Gamma \neq \Sigma$. Let us show that $\Gamma R \Sigma$.

Suppose $\varphi \in \Sigma$. Since $\Gamma \neq \Sigma$, there exists a formula $\psi \in \Sigma$ such that $\psi \notin \Gamma$. By CMT, $M_{\mathbf{DL}}, \Sigma \vDash \varphi \wedge \psi$. Since $\Gamma R \Delta R \Sigma$, it follows that $M_{\mathbf{DL}}, \Gamma \vDash \Diamond\Diamond(\varphi \wedge \psi)$. Again by CMT,

$$M_{\mathbf{DL}}, \Gamma \vDash \Diamond\Diamond(\varphi \wedge \psi) \to \Diamond(\varphi \wedge \psi) \vee (\varphi \wedge \psi);$$

thus $M_{\mathbf{DL}}, \Gamma \vDash \Diamond(\varphi\wedge\psi)\vee(\varphi\wedge\psi)$. However, $\psi \notin \Gamma$ by assumption, so by CMT, $M_{\mathbf{DL}}, \Gamma \nvDash \psi$. Hence $M_{\mathbf{DL}}, \Gamma \nvDash (\varphi \wedge \psi)$. It follows that $M_{\mathbf{DL}}, \Gamma \vDash \Diamond(\varphi \wedge \psi)$, leave alone $M_{\mathbf{DL}}, \Gamma \vDash \Diamond\varphi$. Thus $\Diamond\varphi \in \Gamma$ by CMT.

Eventually we see that $\varphi \in \Sigma$ implies $\Diamond\varphi \in \Gamma$, i.e., $\Gamma R \Sigma$ by definition. □

By Propositions 1, 3 we obtain

Proposition 4 **DL** *is canonical.*

Theorem 10 **DL** *is Kripke-complete.*

Proof. From Lemma 2 and Proposition 4. □

Actually canonicity holds for a large class of modal logics. There is a remarkable result (*Sahlqvist theorem*) stating canonicity for logics with axioms of special form, so-called *Sahlqvist formulas.* Two axioms of the logic **DL**, and also axioms of reflexivity and transitivity are examples of Sahlqvist formulas. In particular, it follows that **S4** is Kripke-complete. Exact formulation and proof of Sahlqvist theorem can be found, e.g., in [CZ97].

Now we need another property of Kripke-complete logics.

Consider a frame $F = (W, R)$. The *restriction of* R *to* $V \subseteq W$ is the relation $R{\upharpoonright}V = R \cap (V \times V)$, i.e., for all $y, z \in V$ we have $y\,(R{\upharpoonright}V)\,z$ iff yRz. For $x \in W$, put $R(x) = \{y \mid xRy\}$; for $V \subseteq W$, put $R(V) = \{y \mid \exists x \in V\ xRy\}$.

The *cone in* F *with a root* x is the frame $(W_x, R{\upharpoonright}W_x)$, where

$$W_x = \{x\} \cup R(x) \cup R(R(x)) \cup \ldots.$$

The following fact is well-known (cf. [CZ97], section 3.3):

Proposition 5 *The logic of a frame is the intersection of the logics of its cones.*

Thus we have:

Proposition 6 *A Kripke-complete logic is the logic of the class of cones of its frames.*

We now return to the logic **DL**. A frame (W, R) such that xRy for all distinct x and y (i.e., $\neq_W \subseteq R$) is called a *cluster.* For example, every frame of form $(X, \neq_X)$ is a cluster. Obviously, clusters are **DL**-frames.

Proposition 7 *Every cone in a* **DL***-frame is a cluster. The other way round, every cluster is a cone (with a root at any of its points).*

Thus we obtain:

Proposition 8 **DL** *is complete with respect to clusters.*

We also need the following construction.

Definition 5 Consider frames $F = (W, R)$, $G = (V, S)$, and a map $f : W \to V$.

f *is monotonic* if for all $x, y \in W$, xRy implies $f(x)Sf(y)$;

f has the *lift property,* if for all $x \in W$, $z \in V$ such that $f(x)Sz$, there exists $y \in R(x)$ such that $f(y) = z$.

A surjective monotonic map with the lift property is called a *p-morphism.* The notation $F \twoheadrightarrow G$ means the there exists a p-morphism from F onto G.

Proposition 9 (p-morphism lemma) *If $F \twoheadrightarrow G$, then $\mathbf{L}(F) \subseteq \mathbf{L}(G)$.*

For the proof see, e.g., [CZ97], section 3.3.

Theorem 11 $\mathbf{DL} = \mathbf{L}_{\neq}$.

Proof. By Proposition 8, **DL** is the logic of the class of clusters. This class incudes all frames of form $(X, \neq_X)$, thus $\mathbf{DL} \subseteq \mathbf{L}_{\neq}$.

Let us check that $\mathbf{L}_{\neq} \subseteq \mathbf{DL}$. We will show that for every cluster $F = (W, R)$ there exists a frame $G = (X, \neq_X)$ such that $G \twoheadrightarrow F$. Let V be the set of all the reflexive points in the frame F, and let V' be another copy of V such that $V' \cap W = \emptyset$. Let i be a bijection from V' onto V. We put $X = W \cup V'$ and define $f : X \to W$ by setting $f(v) = v$ for $v \in W$, and $f(v) = i(v)$ for $v \in V'$. It is straightforward to check that f is a required p-morphism.

Therefore $\mathbf{L}(G) \subseteq \mathbf{L}(F)$, so $\mathbf{L}_{\neq} \subseteq \mathbf{L}(F)$. Since F is an arbitrary cluster, it follows that $\mathbf{L}_{\neq} \subseteq \mathbf{DL}$. $\square$

5.2 Decidability and complexity

Theorem 12 **DL** *is the logic of finite frames of the form $(X, \neq_X)$. Moreover, $\varphi \in \mathbf{DL}$ iff φ is valid on all frames $(X, \neq_X)$ of size less than $2|\varphi|$.*

Proof. Clearly, the logic of finite frames of the form $(X, \neq_X)$ includes **DL**.

To check the converse inclusion, we show that finite clusters falsify all undesirable formulas. Suppose that $\varphi \notin \mathbf{DL}$. By Theorem 11, φ is not valid on some frame $F = (W, \neq_W)$. We shall construct a finite frame $F_0 = (X, \neq_X)$ falsifying φ.

For a valuation V and a point x we have $M, x \vDash \neg\varphi$, where $M = (F, V)$. Let φ' be the result of replacing each $\Box$ in φ with $\neg\Diamond\neg$. Then φ' is equivalent to φ in $\mathbf{K}$, so $M, x \vDash \neg\varphi'$. Now let $\Psi = \{\Diamond\alpha_1, \ldots, \Diamond\alpha_n\}$ be the set of all subformulas of φ' beginning with $\Diamond$. For every α_i we define $V_i \subseteq W$ as the set of all those points in M, where α_i is true. Let U_i be the following subset of V_i: if V_i contains at most two elements, we put $U_i = V_i$; otherwise, we chose arbitrary $a, b \in V_i$ and put $U_i = \{a, b\}$. Put $X = U_1 \cup \cdots \cup U_n \cup \{x\}$. Since n is less than the length of φ, X has less than $2|\varphi|$ elements. For the frame $F_0 = (X, \neq)$ we define the valuation V_0 by setting $V_0(p) = V(p) \cap X$; put $M_0 = (F_0, V_0)$.

By the construction, for every $y \in X$ and every variable p we have

$$M_0, y \vDash p \iff M, y \vDash p.$$

It is not difficult to check that this equivalence extends to every subformula ψ of φ' (by induction on the length of ψ):

$$M_0, y \vDash \psi \iff M, y \vDash \psi.$$

We consider only the less obvious case $\psi = \Diamond\alpha_i$.

If $M_0, y \vDash \Diamond\alpha_i$, then $M_0, y' \vDash \alpha_i$ for some $y' \neq y$; by the induction hypothesis, $M, y' \vDash \alpha_i$; thus, $M, y \vDash \Diamond\alpha_i$. Conversely, assume that $M, y \vDash \Diamond\alpha_i$ and show that $M_0, y \vDash \Diamond\alpha_i$. Note that in this case V_i is non-empty, and so U_i is; moreover, U_i contains a point y' distinct from y. Then $y' \in X$; also $M, y' \vDash \alpha_i$ by the definition of U_i. By the induction hypothesis, $M_0, y' \vDash \alpha_i$, and so $M_0, y \vDash \Diamond\alpha_i$, as required.

Since $M, x \nvDash \varphi'$, from the above equivalence we obtain that $M_0, x \nvDash \varphi'$, so $M_0, x \nvDash \varphi$. It follows that $F_0 \nvDash \varphi$. □

Theorem 13 **DL** *is decidable and moreover,* coNP-*complete*[19]*.*

This theorem is immediate from Theorem 12. The decidability follows from finite axiomatizability of **DL** and completeness with respect to finite frames (see Section 3.3). Moreover, every non-theorem of **DL** is falsified in a finite frame $(X, \neq_X)$ of size $< 2|\varphi|$; to falsify a formula, we need to "guess" the size of X (which is polynomial in length of the formula) and the valuation for variables occurring in the formula.

5.3 The logic of the difference relation on an infinite set

We have shown that **DL** is the logic of the class of all frames of the form $(X, \neq_X)$. Now let us consider the logic of a single infinite frame $(X, \neq_X)$. As we shall see, this logic does not depend on X.

For $n \geq 0$, consider the formula

$$Ad_n := \Diamond p_1 \wedge \ldots \wedge \Diamond p_n \to \Diamond(\Diamond p_1 \wedge \ldots \wedge \Diamond p_n)$$

(as usual, for $n = 0$ the conjunction $\varphi_1 \wedge \ldots \wedge \varphi_n$ is assumed to be $\top$; thus Ad_0 is equivalent to $\Diamond\top$).

Let $\mathcal{C}$ be the class of all finite clusters containing at least one reflexive point.

Theorem 14 *Let X be an infinite set. Then*

(1) $ML(X, \neq_X) = ML(\mathcal{C})$. *Moreover,* $(X, \neq_X) \vDash \varphi$ *iff* φ *is valid on all finite clusters from* $\mathcal{C}$ *of size* $\leq 2|\varphi|$.

(2) $ML(X, \neq_X) = \mathbf{DL} + \{Ad_n \mid n \geq 0\}$.

Corollary 4 *For infinite X, $ML(X, \neq_X)$ does not depend on X.*

---

[19] The decision problem for Y is in the class coNP if the decision problem for the complement of Y is in the class NP; recall that NP is the class of problems decidable in polynomial time by non-deterministic Turing machines. A problem Y is coNP-*hard* if every problem from coNP is polynomial time reducible to Y; a problem is coNP-*complete* if it is in coNP and also coNP-hard. Cf., e.g., [AB09].
Note that the decision problem for any consistent modal logic L is coNP-hard, since the decision problem for classical propositional logic is trivially reducible to decision problem for L.

Corollary 5 *Let X be a non-empty set. Then the validity problem on the frame $(X, \neq_X)$ is decidable and* coNP*-complete.*

To prove Theorem 14, we need several auxiliary facts.

Proposition 10 $(W, R) \vDash Ad_n \;\Leftrightarrow$

$$\forall x \forall y_1 \dots \forall y_n\, (xRy_1 \wedge \dots \wedge xRy_n \;\Rightarrow\; \exists z(xRz \wedge zRy_1 \wedge \dots \wedge zRy_n)).$$

Proof. A simple exercise. □

Put $L = \mathbf{DL} + \{Ad_n \mid n \geq 0\}$.

Proposition 11 *Let F be a cone. Then F is an L-frame iff F is an infinite cluster or a cluster containing a reflexive point.*

Proof. By Proposition 10. □

The logic L (as well as $\mathbf{DL}$) is canonical (e.g., since every Ad_n is a Sahlqvist formula), i.e., $F_L \vDash L$.

Lemma 3 *Let F be a cone in the canonical frame F_L. Then F contains a reflexive point.*

Proof. Assume that $F = (W, R)$ is an infinite cluster. Consider the set of formulas $\Psi = \{\Diamond\varphi \mid \exists x\, (\varphi \in x \in W)\}$ and show that it is consistent in L. Let $\Psi_0 = \{\Diamond\varphi_1, \dots, \Diamond\varphi_n\}$ be a finite subset of Ψ. By definition, there exist points $y_1, \dots, y_n$ such that $\varphi_i \in y_i \in W$, $1 \leq i \leq n$. In an infinite W there exists a point $x \in W$ distinct from $y_1, \dots, y_n$. Since xRy_i for all i, the formula $\Diamond\varphi_1 \wedge \dots \wedge \Diamond\varphi_n$ in true at x in the canonical model; thus Ψ_0 is consistent in L. Hence the consistency of Ψ follows.

Every consistent set can be extended to a maximal one (this statement is called Lindenbaum lemma), so $\Psi \subseteq z$ for some $z \in W_L$. By the definition of the set Ψ (and by the definition of the canonical relation) it follows that zR_Ly for all $y \in W$. Since F is a cluster and R_L is symmetric, we have $z \in W$. Therefore zR_Lz. □

Proof of Theorem 14. Since all axioms of L are valid on the frame $(X, \neq_X)$, we have $L \subseteq ML(X, \neq_X)$.

It is easy to check that $(X, \neq_X) \twoheadrightarrow F$ for any $F \in \mathcal{C}$, thus $ML(X, \neq_X) \subseteq ML(\mathcal{C})$.

Let us check that $ML(\mathcal{C}) \subseteq L$. Assume that $\varphi \notin L$. Then φ is falsified in a cone F of the canonical frame F_L (by CMT and Proposition 5). Thus for some valuation V and some point x we have $(F, V), x \vDash \neg\varphi$. By Lemma 3, there exists a reflexive point y in the frame F. Similarly to the proof of Theorem 12 we can construct a cluster F_0 of cardinality $\leq 2|\varphi|$ containing both x and y, such that φ is not valid on F_0 and y is reflexive in F_0.

Thus $L = ML(X, \neq_X) = ML(\mathcal{C})$, which proves Theorem 14. □

Note that if $|X| = n+1$, then $(X, \neq_X) \vDash Ad_i$ for $i < n$, but $(X, \neq_X) \nvDash Ad_n$. Hence it easily follows that the logic L does not have a finite axiomatisation, that is, for every finite set of axioms Ψ we have $L \neq \mathbf{K} + \Psi$. In fact, a more general theorem holds:

Theorem 15 *If $L = \mathbf{K} + \Psi$, then for every m there exists a formula in Ψ containing more than m distinct variables.*

Proof. Suppose for the sake of contradiction that for some m every formula in Ψ contains at most m distinct variables.

Consider the frame $F = (X, \neq_X)$, where $X = \{0, \dots, 2^m\}$, and the frame $F' = (\{0, \dots, 2^m - 1\}, R)$, where xRy iff ($x \neq y$ or $x = y = 0$). It is not difficult to show that if a formula ψ contains at most m distinct variables, then $F \vDash \psi \;\Leftrightarrow\; F' \vDash \psi$ (the details can be found in [SS05]). The formula Ad_{2^m} is falsified in the frame F, which means that F is not an L-frame. It follows that F falsifies some formula $\varphi \in \Psi$. Since φ contains at most m distinct variables, it is also falsified in F', by the above observation. On the other hand, F' belongs to the class $\mathcal{C}$ from Theorem 14, thus F' validates L containing φ. We have a contradiction. $\square$

Historical remarks

Theorem 11 was proved by K. Segerberg [Seg76] (see also [Seg80]). According to [dR92], the observation on complexity of **DL** was first made in an unpublished paper [dSvEB90].

Results on the logic of infinite sets are more recent. Apparently, Theorem 15 was first proved in [KS10] (see also [KSS12]). As far as we know, Theorem 14 has never been published before; an axiomatization of the logic of $(X, \neq_X)$ was also known to L. Maksimova (a personal communication to the authors). Some syntactic and algebraic properties of extensions of **DL** were considered in recent works [KM10], [Kar12].

5.4 Standard translation for **DL** and **S5**

Using Theorem 11 we can describe a standard translation for the logic **DL**, with R as the difference relation. In this case, the translation $\varphi^{\neq}$ of a modal formula φ is defined as follows:

$$
\begin{aligned}
&p_i^{\neq}(t) := P_i(t),\ \bot^{\neq}(t) := \bot,\ \top^{\neq}(t) := \top,\\
&(\neg\varphi)^{\neq}(t) := \neg\varphi^{\neq}(t),\\
&(\varphi \to \psi)^{\neq}(t) := \varphi^{\neq}(t) \to \psi^{\neq}(t),\\
&(\varphi \wedge \psi)^{\neq}(t) := \varphi^{\neq}(t) \wedge \psi^{\neq}(t),\\
&(\varphi \vee \psi)^{\neq}(t) := \varphi^{\neq}(t) \vee \psi^{\neq}(t),\\
&(\Box\varphi)^{\neq}(t) := \forall x\, (x \neq t \to \varphi^{\neq}(x)),\\
&(\Diamond\varphi)^{\neq}(t) := \exists x\, (x \neq t \wedge \varphi^{\neq}(x)).
\end{aligned}
$$

The Standard translation theorem (Theorem 8) and Theorem 11 yield an embedding of **DL** in classical predicate calculus (with equality):

Corollary 6 $\mathbf{DL} \vdash \varphi \quad \Leftrightarrow \quad \vdash \forall t\, \varphi^{\neq}(t)$.

In a certain sense, **DL** is a "more expressive" version of the well-known Lewis's system **S5** defined as follows:

$$\mathbf{S5} := \mathbf{S4} + p \to \Box\Diamond p.$$

In Kripke semantics three axioms of this logic correspond to reflexivity, transitivity, and symmetry of binary relations:

Proposition 12 $(W, R) \vDash \mathbf{S5} \;\Leftrightarrow\; R$ *is an equivalence relation.*[20]

The logics **S5** and **DL** are similar in many respects. It is easy to see that the cones of **S5** are frames of the form $(X, X \times X)$. The modal operators $\Box$ and $\Diamond$ on these frames are interpreted as the universal and the existential quantifiers, respectively. By constructions similar to those from Sections 5.1 and 5.2, one can prove the following theorem (for details, see, e.g., [CZ97]):

Theorem 16 *(1)* **S5** *is the logic of the class of all frames of the form* $(X, X \times X)$.[21]
(2) $\varphi \in \mathbf{S5}$ *iff* φ *is valid on all finite frames* $(X, X \times X)$ *of size* $\leq |\varphi|$.
(3) **S5** *is decidable and moreover,* coNP*-complete.*

The standard translation for **S5** is presented in the following way:

$$(\Box\varphi)^{\otimes}(t) := \forall x\, \varphi^{\otimes}(x),$$
$$(\Diamond\varphi)^{\otimes}(t) := \exists x\, \varphi^{\otimes}(x),$$

together with preservation of the classical connectives.

From the Standard translation theorem and Theorem 16 we obtain

Corollary 7 $\mathbf{S5} \vdash \varphi \quad \Leftrightarrow \quad \vdash \forall t\, \varphi^{\otimes}(t)$.

We have mentioned that **DL** is more expressive than **S5**. More precisely, **DL** is interpretable in **S5** as follows. For a modal formula φ, define the modal formula $\varphi^{\odot}$:

$$\begin{aligned}
&p^{\odot} := p,\ \bot^{\odot} := \bot,\ \top^{\odot} := \top,\\
&(\neg\varphi)^{\odot} := \neg\varphi^{\odot},\\
&(\varphi \to \psi)^{\odot} := \varphi^{\odot} \to \psi^{\odot},\\
&(\varphi \wedge \psi)^{\odot} := \varphi^{\odot} \wedge \psi^{\odot},\\
&(\varphi \vee \psi)^{\odot} := \varphi^{\odot} \vee \psi^{\odot},\\
&(\Box\varphi)^{\odot} := \Box\varphi^{\odot} \wedge \varphi^{\odot}\\
&(\Diamond\varphi)^{\odot} := \Diamond\varphi^{\odot} \vee \varphi^{\odot}.
\end{aligned}$$

[20] Algebraic models of **S5** are the so-called *monadic algebras*, see [HG98].
[21] This fact and Corollary 7 was first proved by M. Wajsberg (1933).

It is easy to see that $(X, X\times X) \vDash \varphi$ iff $(X, \neq_X) \vDash \varphi^{\odot}$.[22] Now from Theorems 11 and 16 we obtain

Corollary 8 $\mathbf{S5} \vdash \varphi \;\Leftrightarrow\; \mathbf{DL} \vdash \varphi^{\odot}$.

6 Conclusion

In this paper we discuss only some basic properties of propositional modal logics — Kripke completeness, decidability, algorithmic complexity, finite axiomatizability. Many questions are left beyond our consideration, but deserve special studies. We point out that general problems in the field of modal logic are usually nontrivial — at least because there are continuum many modal logics.

Let us give a brief account of the current research.

- Algebraic problems
  As pointed out in section 3.5, propositional modal logics exactly correspond to varieties of modal algebras. Thus logical and algebraic problems are closely related, and many tasks are solved by combination of logical and algebraic methods. For more details see [CZ97], [BdRV01]. Also note that modal algebras (and respectively, modal logics) arise from other algebraic structures related to logic, such as cylindric and relation algebras; cf., e.g., [MV96].
- Algorithmic problems
  In section 2.4 we mentioned algorithmic problems that make sense for axiomatic calculi, the main of them is the decision problem. It turns out that propositional modal logics are typically decidable, but quite often it is hard to prove decidability of particular logics. A related problem is estimating their computational complexity. Numerous particular questions about properties of modal logics can also be put as algorithmic problems. In general this field is rather active, especially because of applications of modal logics in CS. Cf. [BvBW06], chapters 3, 7; [CZ97], chapters 17, 18.
- Correspondence theory
  A modal formula φ *corresponds* to a classical $\mathcal{L}_0$-formula Φ if on any frame the validity of φ is equivalent to the truth of Φ (see section 4). Correspondence theory studies this relation between modal and classical formulas. Typical questions are the following.
  When a modal formula has a classical correspondent, and the other way round? When a correspondent can be constructed by an algorithm? How complex this algorithm can be? A nontrivial model-theoretic technique is used in this field, but still there exist open difficult problems. Cf. [BvBW06], chapter 5.
- General theory of neighbourhood and topological models of modal logics

[22] In fact, the same holds for every two frames (W, R) and $(W, R^{\odot})$, where $R^{\odot}$ is the reflexive closure of R, i.e., $R^{\odot}$ is obtained by adding pairs of form (x, x) to R.

The main idea of neighbourhood (or topological) models is interpreting necessity as truth in some neighbourhood. So modal logic can deal with topological notions such as limit points, continuity, connectedness etc. Cf. [BvBW06], chapters 1, 16; [APHvB07], chapter 5; [Pac17]. Although neighbourhood semantics is a natural generalization of Kripke semantics, its theory is much less developed, and there are more questions than answers in this field. For example, unlike Kripke semantics, here we know very little about the completeness problem. In general, it is unclear, what topological properties are expressible in modal languages and how to axiomatize complicated structures like smooth manifolds, dynamic systems etc.

- Syntactic properties of modal calculi
  These are different kinds of interpolation property, the disjunction property, unification, admissibility of inference rules. Many deep results were obtained in this area, but general problems remain open. Cf., e.g., [CZ97] or [BvBW06], chapter 8.
- The present paper concentrates on modal logics in propositional languages. Investigation of first-order modal logics is a separate large area of active research. Cf. [FM98] for the philosophical aspects, [GSS09] for the mathematical results in this field.

In section 3.2 we pointed out that the origin of 'points' or 'possible worlds' in Kripke models and the meaning of atomic propositions are not essential from the mathematical viewpoint. So the formal approach to semantics leaves room for various applications of modal logic.

Let us mention some areas of these applications.

- Formal verification
  If a Kripke frame is regarded as a collection of states and transitions of a computing system (a 'transition system', in computer science terminology), then modal formulas can describe properties of computation processes. This allows us to apply modal language to problems of model checking, cf. [BvBW06], ch. 17; [CGP99].
- Description logics
  Description logics (or 'ontology languages') were developed for formal representation of facts from specific subject areas (data or knowledge bases). In many cases description languages are closely related to modal languages, so methods of modal logic are successfully applicable. Description logics are already involved in construction of several huge and efficient knowledge bases, cf. [BvBW06], chapter 13.
- Proof theory
  One of the most important applications of modal logics is in mathematical logic itself, namely in proof theory. Modal logics of this kind are called *provability logics*. In section 3 we mentioned a well-known example, the logic **GL**, where necessity is understood as provability in formal arithmetical theories. For more details about provability logics cf. [Boo93], [AB04], and [BvBW06], chapter 16.

- Epistemic logics
  Independently from the knowledge representation field, modal logicians are developing *epistemic logics*, where 'knowledge' and 'belief' are treated as modalities. They are related to provability logics ([BvBW06], chapter 16), and also to systems of information transmission [FMHV03].
- Spatial logics
  Spatial logics are formal systems describing interaction of spatial objects. The latter can be points or subsets of a certain space with a geometric or topological structure, e.g., intervals on a line, domains in space etc. For application of modal logics in this field cf. [APHvB07], chapters 5, 6, 9, 10.

Finally let us emphasize a global problem of modal logic related to its mathematical side and essential for applications — combining modal systems. There are different kinds of combined and extended modal logics developed in recent decades — polymodal logics (such as temporal logics or modal products, cf. [KWZG03]), dynamic modal logics and modal μ-calculus (cf. [BvBW06], chapters 1, 12), graded modal logics (cf. [BvBW06], chapter 4) etc. Logics emerging in these areas successfully combine expressive power with algorithmic properties.

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