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Partitioning Kripke frames of finite height

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Abstract. In this paper we prove the finite model property and decidability of a family of modal logics. A binary relation R is said to be pretransitive if $R^* = \bigcup_{i\leqslant m} R^i$ for some $m \geqslant 0$, where R^* is the transitive reflexive closure of R. By the height of a frame (W, R) we mean the height of the preorder (W, R^*). We construct special partitions (filtrations) of pretransitive frames of finite height, which implies the finite model property and decidability of their modal logics.

Keywords: modal logic, finite model property, decidability, pretransitive relation, finite height.

§ 1. Introduction and main results

The language of propositional modal logic is the language of classical propositional logic with additional connectives. Although it is simple, it turns out to be an effective tool for describing properties of relations. The resulting theories often have better properties (such as, for example, algorithmic decidability or low complexity) than the corresponding first-order theories. This led to the widespread use of modal logic in computer science (see, for example, [1], [2]) and also in other areas of mathematical logic: in the study of fragments of predicate logics, proof theory, set theory and algebraic logic (see, for example, [3]–[6]).

Algebraic and relational models. The first modal calculi arose in the 1930s as a formalization of the concept of possible truth [7]. It soon became clear that modal systems have a natural algebraic interpretation for which logic is the equational theory of a certain algebra: the derivability of a formula in the calculus corresponds to the truth of an identity in the algebra. The central objects turned out to be *modal algebras* that are Boolean algebras with additional additive operations, and the corresponding theories are *modal logics* (more precisely, *normal propositional modal logics*). For example, the closure algebra of a topological space (the Boolean algebra of subsets of the space with the additional unary operation that takes every set to its closure) is a modal algebra.

The study of modal algebras and their theories began in the 1940s in the works of Tarski, McKinsey, Jónsson, and others. In [8], two of the deductive systems described in [7] (S4 and S5) were regarded as theories of closure algebras and of

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monadic algebras; the first results on the decidability of modal logics were proved in [9]. In [10], Jónsson and Tarski proved a representation theorem for modal algebras which is a generalization of Stone's theorem on the representation of Boolean algebras.

In contrast to Boolean algebras having the same equational theory in the non-trivial case (classical propositional logic), modal algebras lead to more varied theories: a continuum of different modal logics already occurs in the case of a single modal operation [11]. In particular, this implies that among the modal logics there are algorithmically undecidable ones.

It turned out that *relational semantics*, or *Kripke semantics*, which was proposed in the late 1950s, is of fundamental importance in the further development of modal logic. A non-empty set with relations on it is called a (*Kripke*) *frame*. We shall mainly be interested in the case when there is a unique binary relation in the frame. Corresponding to such a frame (W, R) is a modal algebra $A(W, R)$, which is the Boolean algebra of all subsets of W with an additional unary operation $\Diamond_R$: for $V \subseteq W$ the set $\Diamond_R(V)$ is the preimage of V under R:

$$\Diamond_R(V) = \{w \mid \exists v \in V \ wRv\}.$$

The modal logic of a frame or of a class of frames is the equational theory of the corresponding algebras.

The interpretation of modal formulae on sets with relations led to the appearance of diverse applications of modal logic. For example, any relational structure is a model of a first-order language. This enables us to view modal logics as fragments of first-order theories; in many cases, such fragments turn out to be decidable [3]. A frame F (with a valuation in $A(F)$) can be regarded as a system of transitions of some computing system. In this case, modal formulae describe properties of the computational process [2]. Modal logics found another important application in proof theory. In the 1930s, Gödel had already suggested the use of modal operators to describe provability and consistency in formal arithmetic [12]. A complete axiomatization of arithmetical provability in the modal language was constructed (much later) by Solovay [13]. In relational semantics, this logic is defined by finite partial orders, which implies its decidability.

The finite model property and decidability. The problem of algorithmic decidability is one of the key questions in the study of modal logics. In many cases, decidability follows from the finite model property (Harrop's theorem). A logic *has the finite model property* if it is the logic of some class of finite frames. The finite model property of modal logics has been systematically studied since the mid-1960s [14]–[16]. It is known that many logics have this property. Logics without it also exist (moreover, there are continuously many such logics; see, for example, Theorem 6.1 in [17]), but they are rather exotic in the case of a single modal operator and are usually constructed artificially. At the same time, there are a number of naturally defined logics for which the question of possessing the finite model property (and of decidability) is open.

Apparently, the most well-known question of this kind is the finite model property and decidability of logics of (m, n)-*frames*, that is, frames in which the relation satisfies the condition $R^n \subseteq R^m$. For example, the transitivity $R \circ R \subseteq R$ is a condition of this kind. In this case ($m = 1$, $n = 2$), as well as in all cases when one of the parameters is less than 2 and in the trivial case $m = n$, the finite model property and decidability have been known since the early 1970s; see [16]. The finite model property and decidability in other cases are old open problems; see Problem 11.2 in [17] and Problem 6 in [18] (in the latter, this task was called '*one of the most intriguing open problems in Modal Logic*'). The answer is not known for any $m, n > 1$, $m \neq n$.

We denote the class of all (m, n)-frames by $\mathcal{F}(m, n)$.

Question 1. *For what values of* m *and* n *does the logic of class* $\mathcal{F}(m, n)$ *have the finite model property*? *When it is decidable*?

Other examples of logics for which the finite model property and decidability are not known are the so-called *pretransitive frames*, that is, frames in which the property of *m-transitivity* holds for some $m \geqslant 0$: $R^* = \bigcup_{i \leqslant m} R^i$. The answer is positive for $m = 0$ and $m = 1$; in other cases, the question is open.

We denote by $\mathcal{G}(m)$ the class of all m-transitive frames.

Question 2. *For what values of* m *does the logic of class* $\mathcal{G}(m)$ *have the finite model property*? *When it is decidable*?

In some cases, the finite model property can be established by constructing special partitions of frames.

Let $\mathcal{B}$ be a partition of a set W and let R be a binary relation on W. We define a relation $R_{\mathcal{B}}$ on the elements of $\mathcal{B}$ by setting for arbitrary $U, V \in \mathcal{B}$

$$U \, R_{\mathcal{B}} \, V \qquad \Leftrightarrow \qquad \exists\, u \in U \quad \exists\, v \in V \quad uRv.$$

The following fact is known: the logic of a class of frames $\mathcal{F}$ turns out to have the finite model property if for every finite partition $\mathcal{A}$ of any frame $(W, R) \in \mathcal{F}$ there is a finite refinement $\mathcal{B}$ of $\mathcal{A}$ such that $(\mathcal{B}, R_{\mathcal{B}}) \in \mathcal{F}$. In this case we say that $\mathcal{F}$ *admits minimal filtrations*.

We now give another sufficient condition for the finite model property. A partition $\mathcal{B}$ of a set W is said to be *tuned* if the following condition holds for any $U, V \in \mathcal{B}$:

$$\exists\, u \in U \quad \exists\, v \in V \quad uRv \qquad \Rightarrow \qquad \forall\, u \in U \quad \exists\, v \in V \quad uRv.$$

A frame is said to be *tunable* if all its finite partitions admit a tuned finite refinement. The logic of every class of tunable frames has the finite model property.

In the case when $\mathcal{F}$ is the class of all the frames of some logic, the second condition is stronger: if all the frames in $\mathcal{F}$ are tunable, then this class admits minimal filtrations.

We note that the above conditions do not require the study of axiomatic or other syntactic properties of the logic in question and enable one to obtain results on the finite model property in a purely semantic way.

Main results. In this paper we construct partitions of pretransitive frames of *finite height*. A partial order has *finite height* h if h is the largest cardinality of chains in this order. To every frame (W, R) one can naturally assign a partial order whose elements are the equivalence classes with respect to the relation

$$w \sim_R v \qquad \Leftrightarrow \qquad wR^*v \quad \text{and} \quad vR^*w,$$

where R^* stands for the transitive reflexive closure of R. These classes are ordered by the relation $\leqslant_R$: $[w] \leqslant_R [v] \ \Leftrightarrow\ wR^*v$. The *height of a frame* is the height of the corresponding partial order.

In Theorem 1 (see § 3) it is established that every frame of finite height in which the number of elements in every equivalence class of $\sim_R$ does not exceed some fixed positive integer N is tunable. This crucial theorem enables us to prove the stability with respect to minimal filtrations of various classes of frames.

We note that when $n > m$ every (m, n)-frame is $(n-1)$-transitive. We have succeeded in constructing the partitions we need for all the frames in the classes $\mathcal{F}(m, n)$ and $\mathcal{G}(m)$, $n > m \geqslant 1$, whose height is finite. Let $\mathcal{F}(m, n, h)$ be all the frames of class $\mathcal{F}(m, n)$ of height not exceeding h, and $\mathcal{F}_*(m, n)$ all the (m, n)-frames of finite height. Similarly, we define the classes of m-transitive frames of finite height $\mathcal{G}(m, h)$ and $\mathcal{G}_*(m)$.

Theorem 2 (see § 4) establishes the finite model property of logics of the classes $\mathcal{F}(m, n, h)$, $\mathcal{G}(m, h)$, $\mathcal{F}_*(m, n)$ and $\mathcal{G}_*(m)$ for $n > m \geqslant 1$ and $h \geqslant 1$.

Corollary. *When $n > m \geqslant 1$ the logic of class $\mathcal{F}(m, n)$ has the finite model property if and only if it coincides with the logic of class $\mathcal{F}_*(m, n)$. A similar criterion holds for the logics of classes $\mathcal{G}(m)$.*

Further results are related to the study of logics of pretransitive frames as deductive calculi.

Theorem 3 (see § 5) is an analogue of Glivenko's theorem for pretransitive logics. We recall that, in the Kripke semantics, the intuitionistic logic is the logic of all partial orders, and the classical logic is the logic of partial orders of height 1. Glivenko's theorem asserts that the deducibility of a formula φ in the classical propositional logic is equivalent to the deducibility of $\neg\neg\varphi$ in the intuitionistic logic. Theorem 3 describes the corresponding reducibility for pretransitive logics.

Theorem 4 (see § 5) describes the modal axiomatics for the class of pretransitive frames of finite height. In particular, it implies the decidability of the logics of $\mathcal{F}(m, n, h)$ and $\mathcal{G}(m, h)$ for all $n > m \geqslant 1$ and $h \geqslant 1$.

The paper is organized as follows. In § 2 we give some preliminary information. In § 3 we construct tuned partitions of frames of finite height. In § 4 we construct partitions of frames of finite height in the classes $\mathcal{F}(m, n)$ and $\mathcal{G}(m)$. In § 5 we consider problems of modal axiomatization. A discussion of the results and some of their consequences are given in § 6.

§ 2. Preliminaries

2.1. Language and semantics. The set of *modal formulae* is constructed from a countable set of variables $PV = \{p_1, p_2, \dots\}$ and constants $\bot$ ('*false*') and $\top$ ('*true*') using the connectives $\wedge$, $\vee$, $\neg$, $\to$ and $\leftrightarrow$ and the unary connective $\Diamond$ ('*diamond*').

Modal logics as deductive systems will be defined in § 5, where axiomatization problems are considered. For the needs of this section and the following two, the notion of the *logic of Kripke frames* is sufficient.

By a (*Kripke*) *frame* we mean a pair (W, R), where W is a non-empty set and $R \subseteq W \times W$. By a *valuation* on a frame we mean a map $\theta\colon PV \to \mathcal{P}(W)$, where $\mathcal{P}(W)$ stands for the set of all subsets of W. A *model* M over a frame $F = (W, R)$ is a triple (W, R, θ), where θ is a valuation on F. The truth value of a modal formula at a point of the model is defined by induction on the length of the formula and is denoted by $M, w \models \varphi$: for a variable p we set $M, w \models p \iff w \in \theta(p)$. Boolean connectives are interpreted in the standard way: for example, $M, w \models \neg\varphi \iff M, w \not\models \varphi$. For the connective $\Diamond$ we set

$$M, w \models \Diamond\varphi \qquad \Leftrightarrow \qquad \exists\, v\,(wRv \quad \text{and} \quad M, v \models \varphi).$$

A formula φ *is true in a model* M if it is true at every point of M, φ *is valid in a frame* F if it is true in every model over F, and φ *is valid in a class of frames* $\mathcal{F}$ if φ is valid in every frame in $\mathcal{F}$. These properties are denoted by the symbols $M \models \varphi$, $F \models \varphi$, and $\mathcal{F} \models \varphi$, respectively. For a set of formulae Φ, the symbol $F \models \Phi$ means that $F \models \varphi$ for all $\varphi \in \Phi$, and then we say that F is a Φ-*frame*. The expressions $M, x \models \Phi$, $M \models \Phi$ and $\mathcal{F} \models \Phi$ are treated in a similar way.

The set of all formulae that are valid in a class $\mathcal{F}$ is denoted by $\operatorname{Log}\mathcal{F}$ and is called the *logic* [1] of $\mathcal{F}$. A set L of formulae is called a *Kripke complete logic* if L is the logic of some class of frames. L has the *finite model property* if L is the logic of some class of finite frames.

2.2. Partitions and filtrations. A formula is said to be *satisfiable in a frame* F (*in a class of frames* $\mathcal{F}$) if it is true at some point of some model over F (over some frame in $\mathcal{F}$).

The finite model property of a Kripke complete logic $\mathrm{L} = \operatorname{Log}\mathcal{F}$ is equivalent to the fact that every formula satisfiable in $\mathcal{F}$ is satisfiable in some finite L-frame. One of the ways to find such a frame is the filtration method.

Definition 1. Consider a frame $F = (W, R)$ and an equivalence relation $\sim$ on W. By the *minimal filtration with respect to* $\sim$ (or the $\sim$-*filtration*) of the frame F we mean the frame $F/{\sim} = (W/{\sim}, R/{\sim})$, where for $U, V \in W/{\sim}$ we set

$$U\, R/{\sim}\, V \qquad \Leftrightarrow \qquad \exists\, u \in U \quad \exists\, v \in V \quad uRv.$$

Definition 2. Let M be a model and φ a formula. We define the equivalence $\sim_\varphi$ *induced by the formula* φ on the points of M as follows: we write $u \sim_\varphi v$ if every subformula of φ is simultaneously true or simultaneously false in u and v.

We say that an equivalence $\sim$ *agrees with a formula* φ *in the model* M if ${\sim} \subseteq {\sim_\varphi}$.

[1] In the introductory section, the logic of frames was defined as an equational theory. Strictly speaking, there is a formal difference between the language of modal formulae and the language of identities of a modal signature. However, this difference is inessential: the validity of a modal formula φ in a frame (W, R) is equivalent to the condition that, in the algebra $A(W, R)$, the value of φ coincides with W, which is the unit of the algebra, for every valuation. The fact that an identity $\varphi = \psi$ holds in the algebra $A(W, R)$ means the validity of the modal formula $\varphi \leftrightarrow \psi$ in the frame (W, R).

We denote the number of subformulae of φ by $l(\varphi)$. Obviously, in every model, the number of classes of $\sim_\varphi$ does not exceed $2^{l(\varphi)}$.

Proposition 3 (lemma on minimal filtrations; see, for example, [19]). *If φ is true at one of the points of a model M over a frame F and the equivalence $\sim$ agrees with φ in M, then φ is satisfiable in $F/{\sim}$.*

The above definition of a minimal filtration is a special case of the more general construction of *a filtration of a Kripke model*. Filtrations arose in the late 1960s in [14] and [15] and later became one of the main tools of proving the finite model property of modal logics [17]. In the case of minimal filtrations, this method reduces to the construction of suitable partitions of frames. As usual, by a (*finite*) *partition of a set W* we mean a (finite) family of pairwise-disjoint non-empty sets whose union coincides with W. If $\mathcal{A}$ and $\mathcal{B}$ are partitions of W and $\forall\, B \in \mathcal{B}\ \exists\, A \in \mathcal{A}\ B \subseteq A$, then $\mathcal{B}$ is called a *refinement of* $\mathcal{A}$. We denote by $\sim_\mathcal{A}$ the equivalence relation whose set of classes coincides with $\mathcal{A}$: $\mathcal{A} = W/{\sim_\mathcal{A}}$. Thus, the points of the frame $F/{\sim_\mathcal{A}}$ are the elements of $\mathcal{A}$. Instead of $F/{\sim_\mathcal{A}}$ and $R/{\sim_\mathcal{A}}$, we write $F_\mathcal{A}$ and $R_\mathcal{A}$.

Definition 4. We say that a class $\mathcal{F}$ of frames *admits minimal filtrations* if for any frame $F \in \mathcal{F}$ and for every finite partition of the domain of F there is a finite refinement of this partition $\mathcal{B}$ such that $F_\mathcal{B} \in \mathcal{F}$. If, moreover, there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that for every frame $F \in \mathcal{F}$ and every finite partition $\mathcal{A}$ of the domain F there is a refinement $\mathcal{B}$ such that $F_\mathcal{B} \in \mathcal{F}$ and

$$|\mathcal{B}| \leqslant f(|\mathcal{A}|),$$

then we say that $\mathcal{F}$ *admits f-bounded minimal filtrations*.

Proposition 5. *If $\mathcal{F}$ admits minimal filtrations, then* $\operatorname{Log}\mathcal{F}$ *has the finite model property. If $\mathcal{F}$ admits f-bounded minimal filtrations, then every formula φ satisfiable in $\mathcal{F}$ is satisfiable in a frame in $\mathcal{F}$ whose size does not exceed $f(2^{l(\varphi)})$.*

Proof. Let φ be satisfiable in $\mathcal{F}$. Then φ is true at one of the points of some model M over some frame $F \in \mathcal{F}$. Consider the equivalence $\sim_\varphi$ on M. Let $\mathcal{A}$ be the equivalence classes of $\sim_\varphi$. The partition $\mathcal{A}$ has a finite refinement $\mathcal{B}$ such that $F_\mathcal{B} \in \mathcal{F}$. The fact that $\mathcal{B}$ is a refinement of $\mathcal{A}$ is equivalent to the fact that $\sim_\mathcal{B}\ \subseteq\ \sim_\mathcal{A}$. By the lemma on minimal filtrations, φ is satisfiable in $F_\mathcal{B}$. If $|\mathcal{B}| \leqslant f(|\mathcal{A}|)$, then, since $|\mathcal{A}| \leqslant 2^{l(\varphi)}$, it follows that the size of $\mathcal{B}$ (that is, the size of the frame thus constructed) does not exceed $f(2^{l(\varphi)})$. This completes the proof of the proposition. $\square$

2.3. Tuned partitions of frames.

Definition 6. Let $F = (W, R)$ be a frame. A partition $\mathcal{A}$ of the set W is said to be *tuned* if the following condition holds for every $U, V \in \mathcal{A}$:

$$\exists\, u \in U \quad \exists\, v \in V \quad uRv \qquad \Rightarrow \qquad \forall\, u \in U \quad \exists\, v \in V \quad uRv.$$

An equivalence relation on W *partitions F tunably* if the set of its equivalence classes is a tuned partition.

A frame is said to be *tunable* if every finite partition of the frame has a tuned finite refinement. A frame is said to be *f-tunable*, where $f\colon \mathbb{N} \to \mathbb{N}$, if each of its finite partitions $\mathcal{A}$ admits a tuned finite refinement $\mathcal{B}$ such that $|\mathcal{B}| \leqslant f(|\mathcal{A}|)$.

The following fact is known (it was apparently noticed for the first time in [20]).

Proposition 7. *If $\mathcal{A}$ is a tuned partition of F, then* $\mathrm{Log}\{F\} \subseteq \mathrm{Log}\{F_{\mathcal{A}}\}$.

Proof. It can readily be seen that, if $\mathcal{A}$ is a tuned partition, then the algebra $A(F_{\mathcal{A}})$ can be embedded in the algebra $A(F)$. □

This fact implies the following assertion.

Proposition 8. *If $\mathcal{F}$ is a class of tunable frames, then the logic* $\mathrm{Log}\,\mathcal{F}$ *has the finite model property. If $\mathcal{F}$ is a class of f-tunable frames, then every formula φ satisfiable in $\mathcal{F}$ is satisfiable in some* $\mathrm{Log}\,\mathcal{F}$*-frame whose size does not exceed* $f(2^{l(\varphi)})$.

Proof. The proof is similar to that of Proposition 5. □

Proposition 9. *Let* $\mathrm{L} = \mathrm{Log}\,\mathcal{F}$ *and let $\mathcal{F}$ be the class of all* L*-frames. If $\mathcal{F}$ consists of tunable frames, then it admits minimal filtrations.*

Proof. This follows from Proposition 7. □

2.4. Frames of finite height. Let R be a binary relation on W. We denote the transitive reflexive closure of R by R^*. We recall that

$$R^0 = Id(W) = \{(x,x) \mid x \in W\}, \qquad R^{i+1} = R^i \circ R, \qquad R^* = \bigcup_{i \geqslant 0} R^i.$$

Consider the frame $F = (W, R)$. We define the following relation on W:

$$x \sim_R y \qquad \Leftrightarrow \qquad xR^*y \quad \text{and} \quad yR^*x.$$

This is an equivalence relation: it is symmetric by definition, and the transitivity and reflexivity are inherited from R^*. The equivalence classes with respect to $\sim_R$ are called the *clusters* of F, and the frame $skF = (W/{\sim_R}, \leqslant_R)$, where

$$[x] \leqslant_R [y] \qquad \Leftrightarrow \qquad xR^*y,$$

is called the *skeleton of F* (here $[z]$ stands for the equivalence class of z with respect to $\sim_R$). The relation $\leqslant_R$ is well defined in this way because, if $x' \sim_R x$ and $y' \sim_R y$, then

$$xR^*y \qquad \Leftrightarrow \qquad x'R^*y'.$$

It can readily be seen that $\leqslant_R$ is a partial order. Moreover, if F is a partial order, then F and skF are isomorphic. We also set $[x] <_R [y] \Leftrightarrow [x] \leqslant_R [y]$ and $[x] \neq [y]$ (in this case, this means that $(y, x) \notin R^*$).

A partial order has *height* h if it contains a chain of cardinality h and there are no chains of larger cardinality. By the *height* $h(F)$ of an arbitrary frame F we mean the height of its skeleton.

We denote by $h(F, x)$ the *depth of a point x in frame F*, that is, the height of the restriction $F{\restriction}\{y \mid xR^*y\}$. We recall that by the *restriction of R to $V \subseteq W$* one means the relation $R{\restriction}V = R \cap (V \times V)$, and by the *restriction of a frame* $F = (W, R)$ to a non-empty subset $V \subseteq W$ one means the frame $F{\restriction}V = (V, R{\restriction}V)$.

The family of points of depth i in F is called the *ith level of F*. Thus, any frame of height h is partitioned into h levels. The clusters corresponding to points of the same level form an antichain in the skeleton of F.

We note that the condition $h(F) = 1$ is equivalent to the condition that R^* is an equivalence relation.

The logics of preorders of finite height are well studied ([21], [22]). In particular, it is known that they have the finite model property. We are interested in the more general case when only the pretranstivity property holds: a frame is said to be *pretransitive* if it is *m-transitive* for some $m \geqslant 0$, that is, it has the property

$$R^* = R^{\leqslant m},$$

where

$$R^{\leqslant m} = \bigcup_{0 \leqslant i \leqslant m} R^i.$$

It can readily be seen that m-transitivity is equivalent to the property

$$R^{m+1} \subseteq R^{\leqslant m}.$$

Pretransitivity follows from many natural properties. Obviously, every finite frame is pretransitive. Moreover, if F is a frame of finite height h and the cardinalities of its clusters are uniformly bounded (that is, there is a positive integer N such that the cardinality of every cluster does not exceed N), then this frame is pretransitive (namely, m-transitive with $m = hN - 1$).

We also note that, if $R^n \subseteq R^m$ for $n > m$, then the frame is $(n-1)$-transitive. Thus, $\mathcal{F}(m, n) \subseteq \mathcal{G}(n-1)$ when $n > m$.

§ 3. Tuned partitions of frames of finite height

Let $\exp_2^i(x)$ denote the tower of exponentials of height i:

$$\exp_2^i(x) = 2^{2^{\cdot^{\cdot^{\cdot^{2^x}}}}} \Big\} i \text{ times}.$$

Theorem 1. *Every frame of finite height in which the cardinalities of the clusters are uniformly bounded by some finite N is tunable. Moreover, if h is the height of such a frame, then this frame is f-tunable, where*

$$f(x) = \exp_2^h\big((N + h + 1)(\log_2 x + N)\big).$$

Proof. Let $F = (W, R)$ be a frame satisfying the conditions of the theorem and let $\mathcal{A}$ be some finite partition of W. We will define, by induction, a sequence of equivalence relations

$$\sim_0 \subseteq \cdots \subseteq \sim_h,$$

tunably partitioning W, and a sequence of the corresponding tuned partitions $\mathcal{B}_i = W/{\sim_i}$. Each of these partitions will be a refinement of $\mathcal{A}$, and the last partition $\mathcal{B}_h$ will turn out to be finite.

We write $h(x)$ instead of $h(F,x)$. We set

$$X_i = \{x \mid h(x) > i\}, \quad Y_i = \{x \mid h(x) = i\}, \quad Z_i = \{x \mid h(x) < i\},$$

that is, Y_i is the ith level of the frame F, Z_i are the points of depth less than i, and X_i are the points of depth exceeding i.

We write

$$\sim_0 = Id(W),$$

that is, $\mathcal{B}_0$ is the partition of W into singletons. This partition is obviously a tuned refinement of $\mathcal{A}$.

Let $1 \leqslant i \leqslant h$. We will define an equivalence relation $\sim_i$. It extends $\sim_{i-1}$ with some pairs of points of the ith level and, at the same time, partitions it into finitely many classes. To this end, we need some auxiliary constructions. We first distinguish the 'upper' part of the partition $\mathcal{B}_{i-1}$:

$$\mathcal{B}'_{i-1} = \{B \in \mathcal{B}_{i-1} \mid B \subseteq Z_i\}.$$

We now fix a signature Ω_i consisting of a single binary predicate symbol, unary predicate symbols $\underline{P}_B$ for all $B \in \mathcal{B}'_{i-1}$, unary predicate symbols $\underline{T}_A$ for all $A \in \mathcal{A}$, and a constant symbol. For every element $u \in Y_i$ we define an $\mathbf{S}(u)$, which is a structure of signature Ω_i. Its domain is the cluster C of F containing the point u. The binary relation on C is the restriction $R{\restriction}C$. For every $B \in \mathcal{B}'_{i-1}$ we define a subset of C interpreting $\underline{P}_B$:

$$P^u_B = \{w \in C \mid \exists\, v \in B\ wRv\}$$

(that is, P^u_B is the intersection of C with the preimage of B under R). For every $A \in \mathcal{A}$, we take $T^u_A = C \cap A$ for the set interpreting $\underline{T}_A$. For the constant, we choose u. Thus,

$$\mathbf{S}(u) = (C, R{\restriction}C,\ (P^u_B)_{B \in \mathcal{B}'_{i-1}},\ (T^u_A)_{A \in \mathcal{A}}, u).$$

On the ith level, we define an equivalence relation $\approx_i$. For $u, v \in Y_i$ we set

$$u \approx_i v \quad \Leftrightarrow \quad \text{the structures } \mathbf{S}(u) \text{ and } \mathbf{S}(v) \text{ are isomorphic.}$$

Finally, we put

$$\sim_i = \sim_{i-1} \cup \approx_i .$$

It can readily be seen by induction that

1) if $u \sim_i v$, then u and v belong to the same level;
2) if $h(u) > i$, then $u \sim_i v \;\Leftrightarrow\; u = v$;
3) if $h(u) < i$, then $u \sim_i v \;\Leftrightarrow\; u \sim_{i-1} v$;
4) if $h(u) = i$, then $u \sim_i v \;\Leftrightarrow\; u \approx_i v$.

Lemma 10. *The partition $\mathcal{B}_i$ is tuned for all $i \leqslant h$.*

Proof. We proceed by induction on i. The partition $\mathcal{B}_0$ is tuned. Let $1 \leqslant i \leqslant h$.

Let $U, V \in \mathcal{B}_i$ and $u_0 R v_0$ for some $u_0 \in U$ and $v_0 \in V$. We consider an arbitrary $u \in U$ and claim that there is a $v \in V$ for which uRv.

If $u_0 \in X_i$, then $U = \{u_0\}$, that is, $u = u_0$, and there is nothing to prove.

If $u_0 \in Z_i$, then U and V are placed in the 'upper' part of the frame Z_i, on which $\sim_i$ coincides with $\sim_{i-1}$. Therefore, $U, V \in \mathcal{B}_{i-1}$. By the induction hypothesis, $\mathcal{B}_{i-1}$ is tuned, and therefore uRv for some $v \in V$.

Finally, let $u_0 \in Y_i$. In this case $u \approx_i u_0$, and therefore there is an isomorphism $g\colon \mathbf{S}(u_0) \to \mathbf{S}(u)$. Let C be the cluster of u_0 and D that of u. Since u_0 is a point of the ith level and $u_0 R v_0$, two cases are possible: $v_0 \in Y_i$ and $v_0 \in Z_i$.

The first case: $v_0 \in Y_i$. In this case, u_0 and v_0 are points of the same level. Since $u_0 R v_0$, it follows that u_0 and v_0 belong to the same cluster, $u_0, v_0 \in C$. We write $v = g(v_0)$ and claim that uRv and $v \in V$. We have $g(u_0) = u$, $g(v_0) = v$ and $u_0 (R{\restriction}C) v_0$, and hence $u(R{\restriction}D)v$, and therefore uRv. Since g is an isomorphism of the structures $\mathbf{S}(u_0)$ and $\mathbf{S}(u)$, and we have $g(v_0) = v$, it follows that g is an isomorphism of $\mathbf{S}(v_0)$ and $\mathbf{S}(v)$. Indeed, the structures corresponding to points of the same cluster differ only by a constant. Therefore, $v_0 \sim_i v$, that is, $v \in V$.

The second case: $v_0 \in Z_i$. In this case, $V \in \mathcal{B}'_{i-1}$, and the signature Ω_i contains the symbol $\underline{P}_V$. Since $u_0 R v_0$, it follows that $u_0 \in P_V^{u_0}$. Since $\mathbf{S}(u_0)$ and $\mathbf{S}(u)$ are isomorphic, we obtain $u \in P_V^u$, that is, uRv for some $v \in V$. $\square$

Lemma 11. *The partition $\mathcal{B}_i$ is a refinement of $\mathcal{A}$ for all $i \leqslant h$.*

Proof. We proceed by induction on i. The partition $\mathcal{B}_0$ is a refinement of $\mathcal{A}$. Let $1 \leqslant i \leqslant h$.

We must verify the inclusion $\sim_i \subseteq \sim_{\mathcal{A}}$. By the induction hypothesis, we have $\sim_{i-1} \subseteq \sim_{\mathcal{A}}$, and therefore it suffices to verify the inclusion $\approx_i \subseteq \sim_{\mathcal{A}}$. Let $u \approx_i v$. For some $A \in \mathcal{A}$ we have $u \in A$. Then $u \in T_A^u$. Since $\mathbf{S}(u)$ and $\mathbf{S}(v)$ are isomorphic, it follows that $v \in T_A^v$, and hence $v \in A$. $\square$

It can readily be proved by induction that all the sets $\mathcal{B}'_0, \dots, \mathcal{B}'_h$ are finite. Indeed, $\mathcal{B}'_0 = \varnothing$ since $Z_1 = \varnothing$. If $i > 0$, then

$$\mathcal{B}'_i = \mathcal{B}'_{i-1} \cup (Y_i/{\approx_i}).$$

If $\mathcal{B}'_{i-1}$ is finite, then the signature Ω_i is finite, and therefore the number of non-isomorphic structures of cardinality at most N in this signature is finite. Hence, the partition $Y_i/{\approx_i}$ is finite.

Since $Z_{h+1} = W$, it follows that $\mathcal{B}'_h = \mathcal{B}_h$, and therefore $\mathcal{B}_h$ is finite. Thus, $\mathcal{B}_h$ is a tuned finite refinement of $\mathcal{A}$, and the first assertion of the theorem is proved.

We now estimate the cardinality of $\mathcal{B}_h$.

Lemma 12. *When $i > 0$,*

$$|Y_i/{\approx_i}| \leqslant N^2 |\mathcal{A}|^N 2^{N|\mathcal{B}'_{i-1}| + N^2}. \tag{1}$$

Proof. To define an Ω_i-structure (up to isomorphism), one must

1) determine the cardinality M of its domain, $1 \leqslant M \leqslant N$;
2) define a binary relation (one of 2^{M^2});

3) define sets interpreting the symbols $\underline{P}_B$, $B \in \mathcal{B}'_{i-1}$ (this gives us $2^{M|\mathcal{B}'_{i-1}|}$ variants);
4) define sets interpreting the symbols $\underline{T}_A$, $A \in \mathcal{A}$ (it can readily be seen that this gives us $|\mathcal{A}|^M$ variants);
5) choose a constant (M variants).

Thus,

$$|Y_i/{\approx_i}| \leqslant N2^{N^2}2^{N|\mathcal{B}'_{i-1}|}|\mathcal{A}|^N N.$$

This completes the proof of the lemma. □

We now estimate the cardinality of $\mathcal{B}_h$. To this end, we estimate the cardinality of $|\mathcal{B}'_i|$ for $1 \leqslant i \leqslant h$. We proceed by induction and show that

$$|\mathcal{B}'_i| \leqslant \exp_2^i\bigl((N+i+1)(\log_2|\mathcal{A}|+N)\bigr).$$

By (1) we obtain

$$|\mathcal{B}'_1| = |Y_1/{\approx_1}| \leqslant N^2|\mathcal{A}|^N 2^{N|\mathcal{B}'_0|+N^2} = N^2|\mathcal{A}|^N 2^{N^2} \leqslant 2^{(N+2)(\log_2|\mathcal{A}|+N)}.$$

Using the induction hypothesis and the inequalities $N \geqslant 1$ and $|\mathcal{A}| \geqslant 1$, we have

$$\begin{aligned}
|\mathcal{B}'_{i+1}| &\leqslant |\mathcal{B}'_i| + |Y_{i+1}/{\approx_{i+1}}| \\
&\leqslant \exp_2^i\bigl((N+i+1)(\log_2|\mathcal{A}|+N)\bigr) + N^2|\mathcal{A}|^N 2^{N|\mathcal{B}'_i|+N^2} \\
&\leqslant \exp_2^i\bigl((N+i+1)(\log_2|\mathcal{A}|+N)\bigr) \\
&\qquad + N^2|\mathcal{A}|^N 2^{N\exp_2^i((N+i+1)(\log_2|\mathcal{A}|+N))+N^2} \\
&\leqslant (N^2|\mathcal{A}|^N+1)2^{N\exp_2^i((N+i+1)(\log_2|\mathcal{A}|+N))+N^2} \\
&\leqslant 2^{2N}|\mathcal{A}|^N 2^{N\exp_2^i((N+i+1)(\log_2|\mathcal{A}|+N))+N^2} \\
&= 2^{N(\log_2|\mathcal{A}|+N+2)+N\exp_2^i((N+i+1)(\log_2|\mathcal{A}|+N))} \\
&\leqslant 2^{\exp_2^i((N+i+1)(\log_2|\mathcal{A}|+N))+N\exp_2^i((N+i+1)(\log_2|\mathcal{A}|+N))} \\
&= 2^{(N+1)\exp_2^i((N+i+1)(\log_2|\mathcal{A}|+N))} \\
&\leqslant 2^{\exp_2^i((N+i+2)(\log_2|\mathcal{A}|+N))} = \exp_2^{i+1}\bigl((N+i+2)(\log_2|\mathcal{A}|+N)\bigr).
\end{aligned}$$

Thus, if x is the cardinality of the original partition $\mathcal{A}$, then the cardinality of the tuned partition $\mathcal{B}_h = \mathcal{B}'_h$ is bounded by $\exp_2^h\bigl((N+h+1)(\log_2 x+N)\bigr)$. This completes the proof of the theorem. □

§ 4. Filtrations in the classes $\mathcal{F}(m,n,h)$ and $\mathcal{G}(m,h)$

We claim that for all $n > m \geqslant 1$ and $h \geqslant 1$ the logics of the classes $\mathcal{F}(m,n,h)$ and $\mathcal{G}(m,h)$ and also of the classes $\mathcal{F}_*(m,n)$ and $\mathcal{G}_*(m)$ have the finite model property. This result is known in the case of height 1: the filtrations for $\mathcal{G}(m,1)$ were constructed in [23], and the proof for the frames $\mathcal{F}(m,n,1)$ was obtained in [24]. Using Theorem 1, we shall extend these results to the case of arbitrary finite height. To this end, one needs to get rid of infinite clusters.

Lemma 13. *Let $\mathcal{A}$ be a finite partition of an (m,n)-frame F, $n > m \geqslant 1$. Then there is a refinement $\mathcal{B}$ of $\mathcal{A}$ such that*

1) *the skeletons of the frames F and $F_{\mathcal{B}}$ are isomorphic*;
2) *all the clusters in $F_{\mathcal{B}}$ are finite and of cardinalities at most $(n-m)|\mathcal{A}|$*;
3) *$F_{\mathcal{B}}$ is an (m,n)-frame.*

Before turning to the construction of partitions of (m,n)-frames, we make several auxiliary observations.

Consider an arbitrary frame $F = (W,R)$. A finite non-empty sequence $w_0 \dots w_n$ of elements of W is called a *path in F* if $w_i R w_{i+1}$ for all $i < n$. The *length of the path* is $l(w_0 \dots w_n) = n$. When $w_0 = w_n$, the path is called a w_0*-loop* (or simply a *loop*). In particular, every sequence of length 1 is a loop of length 0. If the end of a path α coincides with the beginning of a path β, then the *union $\alpha \circ \beta$ of the paths* is the concatenation of the sequences α' and β, where the sequence α' is obtained from α by deleting the last element. In this case, $\alpha \circ \beta$ is a path whose length is the sum of lengths of α and β.

When $w \in W$, let $(G(w), \circ)$ be the monoid of all w-loops. Its identity element is the loop w of length 0. For a w-loop α we set, as usual, $\alpha^0 = w$ and $\alpha^{i+1} = \alpha \circ \alpha^i$.

When $k > 0$, let $g_k(\alpha)$ be the remainder on division of the length of α by k. We write

$$G_k(w) = \{g_k(\alpha) \mid \alpha \in G(w)\}.$$

Proposition 14. *$G_k(w)$ is a subgroup of $\mathbb{Z}_k$ for every $k > 0$, $w \in W$.*

Proof. If α and β are w-loops, then $g_k(\alpha) + g_k(\beta) = g_k(\alpha \circ \beta) \in G_k(w)$, and $g_k(\alpha^{k-1})$ is the inverse of $g_k(\alpha)$. □

Proposition 15. *If $R^* = W \times W$, then*

$$G_k(w) = G_k(v)$$

for every $k > 0$ and $w, v \in W$.

Proof. Let $z \in G_k(v)$, that is, $z = g_k(\gamma)$ for some v-loop γ. We claim that $z \in G_k(w)$. Let α be a path from v to w and β a path from w to v (paths of this kind exist because $R^* = W \times W$). Consider the w-loop

$$\delta = \beta \circ (\alpha \circ \beta)^{k-1} \circ \gamma \circ \alpha.$$

We have $l(\delta) = l(\gamma) + kl(\alpha \circ \beta)$, that is, $g_k(\delta) = z$ and $z \in G_k(w)$. This completes the proof of the proposition. □

When $d > 0$, we define a relation $\preceq_d$ on W as follows:

$$w \preceq_d v \quad \Leftrightarrow \quad \text{there is a path from } w \text{ to } v \text{ whose length is divisible by } d.$$

Obviously, $\preceq_d$ is a preorder on any frame.

Proposition 16. *Let $R^* = W \times W$.*

1) *If d divides the length of every loop, then $\preceq_d$ is an equivalence relation and*
$$|W/{\preceq_d}| \leqslant d.$$

2) *For every $k > 0$ there is a d such that d divides k, d divides the length of every loop, and*
$$\preceq_d \subseteq \preceq_k .$$

Proof. 1) Let us verify the symmetry. Let α be a path from w to v such that d divides $l(\alpha)$. Since $R^* = W \times W$, there is a path β from v to w. The path $\alpha \circ \beta$ is a loop, and therefore d divides $l(\alpha \circ \beta)$. Then d also divides $l(\beta)$.

We claim that $|W/{\preceq_d}| \leqslant d$. Let W contain at least two distinct points. Then there is a path from every point to a different point, and therefore R satisfies the condition
$$\forall\, w \,\exists\, v \;\; wRv.$$
Therefore, the frame contains a path $w_1 \dots w_d$. Every point $w \in W$ is equivalent to one of the points w_i. Indeed, there is a path α from w_d to w. Then $w \preceq_d w_i$ when $i = d - g_d(\alpha)$.

2) Consider a point $u \in W$ and the group $G_k(u)$. If this group is trivial, then we set $d = k$. Otherwise, let d be the least non-zero element of $G_k(u)$. By Proposition 14, this element divides the length of every u-loop. Therefore, by Proposition 15, d divides the length of any loop in the frame.

It remains to construct a path from w to v, which is a multiple of k, in the case when $w \preceq_d v$. If $d \neq k$, then there is a v-loop β such that $g_k(\beta) = d$. Therefore, if α is a path from w to v which is a multiple of d, then, since d divides k, the length of $\alpha \circ \beta^i$ is divisible by k for some i. This completes the proof of the proposition. □

Proposition 17. *Consider a frame $F = (W, R)$ and an equivalence $\sim \subseteq \sim_R$ on F. Let $\overline{F} = (\overline{W}, \overline{R})$ be a $\sim$-filtration of the frame F. Then*

1) *$\overline{W}/{\sim_{\overline{R}}} = \{\overline{C} \mid C \in W/{\sim_R}\}$, where for a cluster $C \in W/{\sim_R}$ we write*
$$\overline{C} = \{D \in W/{\sim} \mid D \subseteq C\};$$

2) *the skeletons of the frames F and $\overline{F}$ are isomorphic.*

Proof. Let $\overline{w}$ denote the $\sim$-class of a point w.

We note that
$$wR^*v \qquad \Leftrightarrow \qquad \overline{w}(\,\overline{R}\,)^*\overline{v}. \tag{2}$$
For if wR^*v, then wR^iv for some i, and $\overline{w}(\,\overline{R}\,)^i\overline{v}$ by the definition of $\overline{R}$. The converse implication holds because $\sim \subseteq \sim_R$.

It follows from (2) that
$$w \sim_R v \qquad \Leftrightarrow \qquad \overline{w} \sim_{\overline{R}} \overline{v}. \tag{3}$$
Therefore, if X is a cluster in $\overline{F}$, then $C = \cup X$ is a cluster in F, and $X = \overline{C}$. If C is a cluster in F, then, by (3) again, $\overline{C}$ is a cluster in $\overline{F}$.

The map $C \mapsto \overline{C}$ is an isomorphism between the skeletons of the frames F and $\overline{F}$ by (2). This completes the proof of the proposition. □

Proof of Lemma 13. Let $\{C_i\}_{i\in I}$ be the clusters of the frame $F = (W, R)$, that is, the equivalence classes with respect to the relation $\sim_R$. We write $R_i = R{\restriction}C_i$. We note that $R_i^* = C_i \times C_i$. We put $k = n - m$. By Proposition 16, there is an equivalence relation $\sim_i$ on C_i such that, if $w \sim_i v$, then k divides the length of some path from w to v and

$$|C_i/{\sim_i}| \leqslant k. \tag{4}$$

Define an equivalence on every cluster C_i as follows:

$$\approx_i = \sim_i \cap \sim_{\mathcal{A}} .$$

By (4) we have

$$|C_i/{\approx_i}| \leqslant k|\mathcal{A}|. \tag{5}$$

Define an equivalence relation $\approx$ on W and the corresponding partition $\mathcal{B}$:

$$\approx = \bigcup_{i\in I} \approx_i, \qquad \mathcal{B} = W/{\approx}.$$

Since $\approx \subseteq \sim_{\mathcal{A}}$, it follows that $\mathcal{B}$ is a refinement of $\mathcal{A}$. Moreover, $\approx \subseteq \sim_R$. Therefore, by Proposition 17, the skeletons of the frames F and $F_{\mathcal{B}}$ are isomorphic, and the cardinalities of the clusters of $F_{\mathcal{B}}$ are bounded by $k|\mathcal{A}|$.

It remains to show that $F_{\mathcal{B}}$ is an (m, n)-frame.

We note that if $[u]_\approx R_{\mathcal{B}} [v]_\approx$, then $uR^{zk+1}v$ for some $z \geqslant 0$. Indeed, by the definition of $R_{\mathcal{B}}$, there are $u' \approx u$ and $v' \approx v$ such that $u'Rv'$. Since $u' \sim_i u$ and $u' \sim_j u$ for some $i, j \in I$, it follows that there are paths α from u to u' and β from v to v' whose lengths are divisible by k.

Let $[u]_\approx (R_{\mathcal{B}})^n [v]_\approx$. Then $uR^{zk+n}v$ for some $z \geqslant 0$. Since F is an (m, n)-frame, it follows that $R^{zk+n} \subseteq \cdots \subseteq R^n \subseteq R^m$. Therefore, $uR^m v$. Hence, $[u]_\approx (R_{\mathcal{B}})^m [v]_\approx$. ☐

Here is an analogue for m-transitive frames of the lemma proved above.

Lemma 18. *Let $\mathcal{A}$ be a finite partition of an m-transitive frame F, $m \geqslant 1$. Then there is a refinement $\mathcal{B}$ of $\mathcal{A}$ such that*

1) *the skeletons of the frames F and $F_{\mathcal{B}}$ are isomorphic;*
2) *all the clusters in $F_{\mathcal{B}}$ are finite and of cardinalities at most $|\mathcal{A}|$;*
3) *$F_{\mathcal{B}}$ is m-transitive.*

Proof. The proof of this fact is simpler than that of the previous lemma. For the filtration, we need not construct a path of a given length. It suffices to show that every path of length exceeding m can be shortened to a path of length not exceeding m.

Let us define $\{C_i\}_{i\in I}$ and R_i in the same way as in the proof of Lemma 13. The partition $\mathcal{B}$ is defined more simply:

$$\approx = \sim_{\mathcal{A}} \cap \sim_R, \qquad \mathcal{B} = W/{\approx}.$$

By the arguments used in the proof of Lemma 13, $\mathcal{B}$ is a refinement of $\mathcal{A}$, the skeletons of the frames F and $F_{\mathcal{B}}$ are isomorphic, and the cardinality of the clusters of $F_{\mathcal{B}}$ are bounded by $|\mathcal{A}|$.

It remains to show that $F_{\mathcal{B}}$ is m-transitive. Since $\approx \subseteq \sim_R$, from $[u]_{\approx} R_{\mathcal{B}} [v]_{\approx}$ one can infer that $uR^{z+1}v$ for some $z \geqslant 0$. Thus, if $[u]_{\approx}(R_{\mathcal{B}})^{m+1}[v]_{\approx}$, then $uR^{m+1+z}v$ for some $z \geqslant 0$. Since F is m-transitive, it follows that $uR^z v$ for some $z \leqslant m$. Hence, $[u]_{\approx}(R_{\mathcal{B}})^z[v]_{\approx}$. □

Theorem 2. *Let $n > m \geqslant 1$ and $h \geqslant 1$. Then the logics of the classes*

$$\mathcal{F}(m,n,h), \quad \mathcal{F}_*(m,n), \quad \mathcal{G}(m,h), \quad \mathcal{G}_*(m)$$

have the finite model property.

Proof. Let φ be satisfiable in some frame $F \in \mathcal{F}_*(m,n)$ and let h be the height of F. Then φ is true at one of the points of some model M over F. Consider the equivalence $\sim_\varphi$ on M. Let $\mathcal{A}$ be the equivalence classes of $\sim_\varphi$. Applying Lemma 13, we obtain a frame $G \in \mathcal{F}_*(m,n)$ of finite height h in which the formula φ is also satisfiable. Moreover, all the clusters in G are finite and uniformly bounded, and therefore, applying Theorem 1 to the partition $\mathcal{B}$ corresponding to the equivalence $\sim_\varphi$ in an appropriate model over G, we obtain a tuned partition $\mathcal{C}$ of the frame G such that φ is satisfiable in $G_{\mathcal{C}}$. The finite model property of the logics of the classes $\mathcal{F}(m,n,h)$ and $\mathcal{F}_*(m,n)$ now follows from Proposition 7.

The proof of the theorem for the classes $\mathcal{G}(m,h)$ and $\mathcal{G}_*(m)$ is completely analogous. One must only use Lemma 18 instead of Lemma 13. This completes the proof of the theorem. □

§ 5. Modal axiomatization

5.1. Definability. We write $\Box\varphi = \neg\Diamond\neg\varphi$,

$$\Diamond^0\varphi = \varphi, \qquad \Diamond^{i+1}\varphi = \Diamond\Diamond^i\varphi, \qquad \Diamond^{\leqslant m}\varphi = \bigvee_{i=0}^{m} \Diamond^i\varphi,$$

$$\Box^0\varphi = \varphi, \qquad \Box^{i+1}\varphi = \Box\Box^i\varphi, \qquad \Box^{\leqslant m}\varphi = \bigwedge_{i=0}^{m} \Box^i\varphi.$$

We can readily verify the following facts.

Proposition 19.
1) (W,R) *is an m-transitive frame* $\Leftrightarrow$ $(W,R) \models \Diamond^{m+1}p \to \Diamond^{\leqslant m}p$.
2) (W,R) *is an (m,n)-frame* $\Leftrightarrow$ $(W,R) \models \Diamond^n p \to \Diamond^m p$.

It is well known that for preorders the property $h(F) \leqslant h$ can be expressed by a modal formula; see, for example, [17], Proposition 3.44. Namely, define (by induction) formulae B_h, $h > 0$, as follows:

$$B_1 = p_1 \to \Box\Diamond p_1, \qquad B_{h+1} = p_{h+1} \to \Box(\Diamond p_{h+1} \vee B_h).$$

For every preorder F,

$$F \models B_h \qquad \Leftrightarrow \qquad h(F) \leqslant h.$$

We describe correspondingr formulae for m-transitive frames. Let $B_h(m)$ denote the formula B_h written using the operators $\Box^{\leqslant m}$ and $\Diamond^{\leqslant m}$:

$$B_1(m) = p_1 \to \Box^{\leqslant m}\Diamond^{\leqslant m} p_1, \qquad B_{h+1}(m) = p_{h+1} \to \Box^{\leqslant m}(\Diamond^{\leqslant m} p_{h+1} \vee B_h(m)).$$

Obviously, in every model over an m-transitive frame we have

$$M, x \models \Diamond^{\leqslant m}\varphi \quad \Leftrightarrow \quad \exists\, y(xR^* y \ \text{ and } \ M, y \models \varphi),$$
$$M, x \models \Box^{\leqslant m}\varphi \quad \Leftrightarrow \quad \forall\, y(xR^* y \Rightarrow M, y \models \varphi).$$

This implies the following assertion.

Proposition 20. *Let F be an m-transitive frame. Then*

$$F \models B_h(m) \qquad \Leftrightarrow \qquad h(F) \leqslant h.$$

A class of frames $\mathcal{F}$ is said to be *modally definable by a set of formulae* Φ if

$$F \in \mathcal{F} \qquad \Leftrightarrow \qquad F \models \Phi.$$

Proposition 21. *Let $m, n, h \geqslant 1$. The class $\mathcal{G}(m, h)$ is modally definable by the formulae $\{\Diamond^{m+1} p \to \Diamond^{\leqslant m} p,\ B_h(m)\}$. When $n > m$, the class $\mathcal{F}(m, n, h)$ is modally definable by the formulae $\{\Diamond^n p \to \Diamond^m p,\ B_h(n-1)\}$.*

Proof. The proof follows from Propositions 19 and 20. □

5.2. Pretransitive logics. A set L of modal formulae is called a *logic* (to be more precise, a *propositional normal modal logic*) if

1) L contains all propositional tautologies;
2) L contains the formulae $\neg\Diamond\bot$ and $\Diamond(p_1 \vee p_2) \to \Diamond p_1 \vee \Diamond p_2$;
3) L is closed with respect to the Modus Ponens rule, the substitution rule, and the *monotonicity rule*, that is,[2]

$$\varphi \to \psi \in \mathrm{L} \qquad \Rightarrow \qquad \Diamond\varphi \to \Diamond\psi \in \mathrm{L}.$$

The smallest logic is denoted by K. If L is a logic and Φ a set of formulae, then $\mathrm{L} + \Phi$ denotes the smallest logic containing $\mathrm{L} \cup \Phi$.

The logic of a class of frames is a logic in the sense of this definition. We recall [17] that $\mathrm{S4} = \mathrm{K} + \{p \to \Diamond p,\ \Diamond\Diamond p \to \Diamond p\}$ is the logic of the class of all (finite) preorders and $\mathrm{S5} = \mathrm{S4} + \{B_1\}$ is the logic of the class of (finite) frames in which the relation is an equivalence.

It follows from Sahlqvist's theorem (see, for example, Theorem 10.30 in [17]) that the following assertion holds.

Proposition 22. *For all $m, n \geqslant 0$ we have* $\operatorname{Log}\mathcal{F}(m, n) = \mathrm{K} + \{\Diamond^n p \to \Diamond^m p\}$ *and* $\operatorname{Log}\mathcal{G}(m) = \mathrm{K} + \{\Diamond^{m+1} p \to \Diamond^{\leqslant m} p\}$.

[2]The definition in which the formulae $\neg\Diamond\bot$ and $\Diamond(p_1 \vee p_2) \to \Diamond p_1 \vee \Diamond p_2$ are replaced by the formula $\Box(p_1 \to p_2) \to (\Box p_1 \to \Box p_2)$ and the monotonicity rule is replaced by the rule $\varphi \in \mathrm{L} \ \Rightarrow\ \Box\varphi \in \mathrm{L}$ is more common. We use different definition because, in the modal language under consideration, $\Diamond$ is a base connective and $\Box$ is defined as an abbreviation. It can readily be seen that these definitions are equivalent (see, for example, Remark 4.7 in [25]).

Definition 23. A logic is said to be *pretransitive* if it contains the *formula of m-transitivity* $\Diamond^{m+1}p \to \Diamond^{\leqslant m}p$ for some $m \geqslant 0$. The symbol trans(L) denotes the least m with this property.

By Proposition 19, all the logics $\mathrm{Log}\,\mathcal{G}(m)$, $m \geqslant 0$, are pretransitive.

Proposition 24. *Let* $m, n \geqslant 0$ *and* $\mathrm{L} = \mathrm{Log}\,\mathcal{F}(m,n)$. *Then*
1) L *is pretransitive and* $\mathrm{trans}(\mathrm{L}) = n-1$ *for* $n > m$;
2) L *is not pretransitive when* $n \leqslant m$.

Proof. If $(W,R) \in \mathcal{F}(m,n)$ and $n > m$, then $R^n \subseteq R^{\leqslant n-1}$. Therefore, L is pretransitive and $\mathrm{trans}(\mathrm{L}) \leqslant n-1$. On the other hand, the frame $(\{1,\dots,n\}, R)$, where $iRj \Leftrightarrow j = i+1$, is an (m,n)-frame, because $R^n = \varnothing$. However, it is not l-transitive for any $l < n-1$.

To prove the second part, we consider a frame $(\mathbb{N}, R)$, where

$$R = \{(i,i+1) \mid i \in \mathbb{N}\} \cup \{(i,i) \mid i \in \mathbb{N}\}.$$

By the reflexivity, the length of any path can be arbitrarily extended in this frame, and therefore this is an (m,n)-frame when $n \leqslant m$. At the same time, this frame is not l-transitive for any l. This completes the proof of the proposition. □

5.3. Glivenko's theorem and the finite model property. Glivenko's theorem asserts that the derivability of a formula φ in the classical propositional logic is equivalent to the derivability of $\neg\neg\varphi$ in the intuitionistic logic. The corresponding reducibility holds for S4 and S5 [26]:

$$\varphi \in \mathrm{S5} \qquad \Leftrightarrow \qquad \Diamond\Box\varphi \in \mathrm{S4}.$$

In Kripke semantics, intuitionistic logic is the logic of all partial orders, and the classical logic is the logic of partial orders of height 1 (see, for example, [17]). Similarly, S4 is the logic of preorders and S5 is the logic of equivalence relations, that is, of preorders of height 1. We now formulate an analogue of Glivenko's theorem for pretransitive logics.

Let L be a pretransitive logic and let $m = \mathrm{trans}(\mathrm{L})$. We write

$$\mathrm{L}[h] = \mathrm{L} + \{B_h(m)\}.$$

We also write $\Diamond^*\varphi = \Diamond^{\leqslant m}\varphi$ and $\Box^*\varphi = \Box^{\leqslant m}\varphi$.

Theorem 3. *If* L *is a pretransitive logic, then*

$$\varphi \in \mathrm{L}[1] \qquad \Leftrightarrow \qquad \Diamond^*\Box^*\varphi \in \mathrm{L}.$$

Proof. Let ψ be a formula and let ψ^* denote the result of replacing $\Diamond$ by $\Diamond^*$ in ψ (resp. replacing $\Box$ by $\Box^*$). It is known that the set $\{\psi \mid \psi^* \in \mathrm{L}\}$ is a logic and contains S4 (see, for example, [27]).

Let $\Diamond^*\Box^*\varphi \in \mathrm{L}$. Then $\Diamond^*\Box^*\varphi \in \mathrm{L}[1]$. Since $p \to \Box^*\Diamond^*p \in \mathrm{L}[1]$, it follows that $\Diamond^*\Box^*p \to p \in \mathrm{L}[1]$. Hence, $\Diamond^*\Box^*\varphi \to \varphi \in \mathrm{L}[1]$ and $\varphi \in \mathrm{L}[1]$.

We carry out the proof in the reverse direction by induction on the length of a derivation of φ in L[1].

Suppose that $\varphi = p \to \Box^*\Diamond^* p$. The formula $\Diamond\Box(p \to \Box\Diamond p)$ is valid in every finite preorder, and hence $\Diamond\Box(p \to \Box\Diamond p) \in$ S4, and therefore $\Diamond^*\Box^*\varphi \in$ L.

The case in which φ is obtained as a result of applying the substitution rule is trivial. Consider the cases of applying the Modus Ponens rule and the monotonicity.

Let ψ, $\psi \to \varphi \in$ L[1] for some ψ. By the induction hypothesis, $\Diamond^*\Box^*\psi$ and $\Diamond^*\Box^*(\psi \to \varphi) \in$ L. Then $\Box^*\Diamond^*\Box^*(\psi \to \varphi) \in$ L (Lemma 1.3.45 of [27]). The formula $\Diamond\Box p \wedge \Box\Diamond\Box(p \to q) \to \Diamond\Box q$ is derivable in S4 since it is valid in every preorder. Therefore, $\Diamond^*\Box^*\varphi \in$ L.

Suppose that $\varphi = \Diamond\psi_1 \to \Diamond\psi_2$ and $\psi_1 \to \psi_2 \in$ L[1]. By the induction hypothesis, $\Diamond^*\Box^*(\psi_1 \to \psi_2) \in$ L. We claim that $\Diamond^*\Box^*(\Diamond\psi_1 \to \Diamond\psi_2) \in$ L. Let $m = \mathrm{trans}(\mathrm{L})$. By Proposition 22, $\mathrm{Log}\, G(m) \subseteq$ L. The formula

$$\Diamond^{\leqslant m}\Box^{\leqslant m}(\psi_1 \to \psi_2) \to \Diamond^{\leqslant m}\Box^{\leqslant m}(\Diamond\psi_1 \to \Diamond\psi_2)$$

is valid in every m-transitive frame and therefore belongs to L. Hence, $\Diamond^{\leqslant m}\Box^{\leqslant m}\varphi \in$ L. This completes the proof of the theorem. □

Recall the notion of a generated submodel [17]. Let $F = (W, R)$ be a frame and let $M = (F, \theta)$ be a model. By the *restriction of* M to $V \neq \varnothing$ we mean the model $M{\restriction}V = (F{\restriction}V, \theta')$, where $\theta'(p) = \theta(p) \cap V$ for all $p \in PV$. When $V \subseteq W$ we say that $R(V) = \{y \mid \exists x \in V\ xRy\}$ is the *image of* V *under* R. If $R(V) \subseteq V$, then $F{\restriction}V$ and $M{\restriction}V$ are called a *generated subframe of* F and a *generated submodel of* M, respectively. The following assertion holds.

Proposition 25 (generated submodel lemma). *Let* $F = (W, R)$, *let* $M = (F, \theta)$, *let* V *be a non-empty subset of* W, *and let* $R(V) \subseteq V$. *In this case,*

1) *if* $x \in V$, *then* $M, x \models \varphi \ \Leftrightarrow\ M{\restriction}V, x \models \varphi$;
2) *if* $F \models \varphi$, *then* $F{\restriction}V \models \varphi$.

Theorem 3 enables us to formulate the following necessary conditions for the decidability and the finite model property of pretransitive logics.

Corollary 26. *Let* L *be a pretransitive logic. If* L *is decidable, then* L[1] *is decidable. If* L *has the finite model property, then* L[1] *has the finite model property.*

Proof. The first assertion follows immediately from Theorem 3.

Let L have the finite model property. We claim that L[1] is the logic of the class of all finite L-frames of height 1. Let $\varphi \notin$ L[1]. Then $\Diamond^*\Box^*\varphi \notin$ L by Theorem 3. Therefore, $\Box^*\Diamond^*\neg\varphi$ is true at one of the points of some model M over some finite L-frame F. Then $\neg\varphi$ is true at some point x of one of the maximal clusters C of M. By the generated submodel lemma, $M{\restriction}C, x \models \neg\varphi$ and $F{\restriction}C$ is an L-frame. It remains to note that $F{\restriction}C$ is a frame of height 1. □

Proposition 27. *Let* L *be a pretransitive logic. Then*

1) L[1] $\supseteq$ L[2] $\supseteq$ L[3] $\supseteq \cdots \supseteq$ L;
2) *if* L *is consistent, then so is* L[1] (*and hence so are all logics* L[h], $h \geqslant 1$).

Proof. The inclusion $\mathrm{L} \subseteq \mathrm{L}[h]$ is obvious. To prove the inclusion $\mathrm{L}[h+1] \subseteq \mathrm{L}[h]$, let $m = \mathrm{trans}(\mathrm{L})$. Since $B_h(m) \in \mathrm{L}[h]$, we have $\Diamond^{\leqslant m} p_{h+1} \vee B_h(m) \in \mathrm{L}[h]$. Then $\Box^{\leqslant m}(\Diamond^{\leqslant m} p_{h+1} \vee B_h(m)) \in \mathrm{L}[h]$, which implies that

$$B_{h+1}(m) = p_{h+1} \to \Box^{\leqslant m}(\Diamond^{\leqslant m} p_{h+1} \vee B_h(m)) \in \mathrm{L}[h].$$

Let L be consistent. We claim that L[1] is consistent. The formula $\neg\Diamond\Box\bot$ is valid in every partial order, and therefore belongs to S4. Hence, $\neg\Diamond^{\leqslant m}\Box^{\leqslant m}\bot \in \mathrm{L}$. Therefore, $\Diamond^{\leqslant m}\Box^{\leqslant m}\bot \notin \mathrm{L}$. By Theorem 3, it is true that $\bot \notin \mathrm{L}[1]$. This completes the proof of the proposition. □

5.4. Kripke completeness of logics of finite height. We shall formulate a sufficient condition for the Kripke completeness of logics containing formulae of finite height.

Recall the definition of the *canonical model of a consistent logic* L (see, for example, [25]). A set Γ of formulae is said to be L-*inconsistent* if $\neg \bigwedge \Gamma_0 \in \mathrm{L}$ for some finite $\Gamma_0 \subseteq \Gamma$. A set Γ is said to be L-*maximal* if it is L-consistent and every proper extension of Γ turns out to be L-inconsistent. It is well known that every L-consistent set is contained in an L-maximal consistent set (Lindenbaum's lemma). By the *canonical frame* of a consistent logic L we mean the frame $F_{\mathrm{L}} = (W_{\mathrm{L}}, R_{\mathrm{L}})$, where W_{L} is the set of all L-maximal sets and R_{L} is defined as follows:

$$\Gamma_1 R_{\mathrm{L}} \Gamma_2 \qquad \Leftrightarrow \qquad \{\Diamond\varphi \mid \varphi \in \Gamma_2\} \subseteq \Gamma_1.$$

The *canonical model* M_{L} is the canonical frame with the valuation θ_{L}, where $\theta_{\mathrm{L}}(p) = \{\Gamma \in W_{\mathrm{L}} \mid p \in \Gamma\}$. The following important fact is well known.

Proposition 28 (canonical model theorem). *If* L *is a consistent logic*, *then*

1) $\forall \Gamma \in W_{\mathrm{L}}\ (M_{\mathrm{L}}, \Gamma \models \varphi \ \Leftrightarrow\ \varphi \in \Gamma)$;
2) $\mathrm{L} = \{\varphi \mid M_{\mathrm{L}} \models \varphi\}$.

A logic is said to be *canonical* if it is valid in its canonical frame. By the canonical model theorem, every canonical logic L is Kripke complete: $\mathrm{L} = \mathrm{Log}\{F_{\mathrm{L}}\}$.

Proposition 29. *Suppose that* $F = (W, R)$ *is the canonical frame of a logic* L. *Then*

1) *for every* $i \geqslant 0$ *we have*

$$xR^i y \qquad \Leftrightarrow \qquad \forall\varphi(\varphi \in y \ \Rightarrow\ \Diamond^i\varphi \in x);$$

2) *if* L *is* m-*transitive*, *then*

$$xR^* y \qquad \Leftrightarrow \qquad \forall\varphi(\varphi \in y \ \Rightarrow\ \Diamond^{\leqslant m}\varphi \in x).$$

Proof. The proof of the first part can be found in Corollary 5.10 of [17]. Let us prove the second part.

Necessity. By Sahlqvist's theorem we have $F \models \Diamond^{m+1}p \to \Diamond^{\leqslant m}p$, and therefore it follows from xR^*y that xR^iy for some $i \leqslant m$. If $\varphi \in y$ in this case, then, by the first part, $\Diamond^i\varphi \in x$, and therefore $\Diamond^{\leqslant m}\varphi \in x$.

Sufficiency. Suppose that xR^*y is false. Then for every $i \leqslant m$ it is false that xR^iy. By the first part, for every $i \leqslant m$ there is a formula $\varphi_i \in y$ such that $\Diamond^i\varphi_i \notin x$. Let $\varphi = \varphi_0 \wedge \cdots \wedge \varphi_m$. It is true that $\varphi \in y$ and $\Diamond^i\varphi \notin x$ for all $i \leqslant m$. Then x contains the formula $\neg\varphi \wedge \neg\Diamond\varphi \wedge \cdots \wedge \neg\Diamond^m\varphi$. The latter means that $\Diamond^{\leqslant m}\varphi \notin x$. This completes the proof of the proposition. □

The following fact results immediately from the definitions.

Proposition 30. *If* L_1 *and* L_2 *are consistent logics and* $\mathrm{L}_2 \subseteq \mathrm{L}_1$, *then the canonical model of* L_1 *is a generated submodel of the canonical model of* L_2 *and consists of all* L_2*-maximal sets containing* L_1.

We denote by $F\langle x\rangle$ the *frame* (*model*) *generated by a point* x: $F\langle x\rangle = F\restriction\{y \mid xR^*y\}$, where R is the relation in the frame (model).

The Kripke completeness and the fact that the extensions of S4 by the formulae B_h are canonical have been well known for quite some time ([21], [22]). The latter fact has the following generalization.

Theorem 4. *If* L *is a pretransitive canonical logic*, *then the logic* $\mathrm{L}[h]$ *turns out to be canonical for all* $h \geqslant 1$.

Proof. Let $M_h = (W_h, R_h, \theta_h)$ denote the canonical model of the logic $\mathrm{L}[h]$ and let $M = (W, R, \theta)$ be the canonical model of L.

By Propositions 27 and 30,

$$M_1 \sqsubseteq M_2 \sqsubseteq M_3 \sqsubseteq \cdots \sqsubseteq M,$$

where $N \sqsubseteq M$ means that N is a generated submodel of M.

Let F be the canonical frame of L.

Lemma 31. *For all* x

$$M, x \models \mathrm{L}[h] \qquad \Leftrightarrow \qquad h(F, x) \leqslant h. \tag{6}$$

Proof. If $h(F, x) \leqslant h$, then $F\langle x\rangle \models \mathrm{L}[h]$ by Proposition 20. Hence, $M\langle x\rangle, x \models \mathrm{L}[h]$. By the generated submodel lemma, $M, x \models \mathrm{L}[h]$. We carry out the proof of the fact that $M, x \models \mathrm{L}[h]$ implies that $h(F, x) \leqslant h$ by induction on the height h.

Let $m = \mathrm{trans}(\mathrm{L})$. Suppose that $M, x \models \mathrm{L}[1]$. Then $x \in W_1$ by Proposition 30. Next, $B_1(m)$ is a Sahlqvist formula, and hence $(W_1, R_1) \models B_1(m)$, which means that $h(W_1, R_1) = 1$ by Proposition 20. Thus, $h(F, x) = 1$.

Now suppose that $M, x \models \mathrm{L}[h+1]$. When $h(F, x) = 1$, there is nothing to prove. We assume that there is a y in the model M_{h+1} such that $[x] <_R [y]$ (here $[x]$ stands for the cluster of a point x in the frame (W, R) or, equivalently, in the frame (W_{h+1}, R_{h+1})). Consider the model $M\langle y\rangle$. We claim that $M\langle y\rangle \models \mathrm{L}[h]$. For this, it suffices to show that, at every point z of the model $M\langle y\rangle$, all substitution instances of the formula $B_h(m)$ are true. Let $\varphi = B_h(m)(\psi_1, \ldots, \psi_n)$. Since $[x] <_R [y]$ and $[y] \leqslant_R [z]$, it is false that $[z] \leqslant_R [x]$. Hence, it is false that zR^*x. By Proposition 29, the last formula means that there is a formula ψ such that $\psi \in x$ and $\Diamond^{\leqslant m}\psi \notin z$. The formula

$$\alpha = \psi \to \Box^{\leqslant m}(\Diamond^{\leqslant m}\psi \vee \varphi)$$

is a substitution instance of the formula $B_{h+1}(m)$, which implies that $M, x \models \alpha$. Hence,

$$M, x \models \Box^{\leqslant m}(\Diamond^{\leqslant m}\psi \vee \varphi).$$

This yields

$$M, z \models \Diamond^{\leqslant m}\psi \vee \varphi.$$

Since $M, z \not\models \Diamond^{\leqslant m}\psi$, it follows that $M, z \models \varphi$, as was to be proved.

Thus, $M\langle y\rangle \models \mathrm{L}[h]$ and, by the induction hypothesis, $h(F, y) \leqslant h$. This implies that $h(F, x) \leqslant h + 1$ and completes the proof of the lemma. □

By Proposition 30, $M, x \models \mathrm{L}[h] \iff x \in W_h$, which shows (by the lemma just proved) that $x \in W_h \iff h(F, x) \leqslant h$. This means that the height of the frame (W_h, R_h) does not exceed h. By Proposition 20, $(W_h, R_h) \models \mathrm{L}[h]$. This proves the theorem. □

Corollary 32. *For $n > m \geqslant 1$ and $h \geqslant 1$ we have*

$$\operatorname{Log}\mathcal{G}(m, h) = K + \{\Diamond^{m+1}p \to \Diamond^{\leqslant m}p,\ B_h(m)\},$$
$$\operatorname{Log}\mathcal{F}(m, n, h) = K + \{\Diamond^{n}p \to \Diamond^{m}p,\ B_h(n-1)\}.$$

Proof. By Sahlqvist's theorem, the logics $K + \{\Diamond^{m+1}p \to \Diamond^{\leqslant m}p\}$ and $K + \{\Diamond^{n}p \to \Diamond^{m}p\}$ are canonical. By Theorem 4, their extensions by formulae of finite height are Kripke complete. By Proposition 21, $\mathcal{G}(m, h)$ is the class of all frames of the first logic and $\mathcal{F}(m, n, h)$ is the class of all frames of the second. □

For a finitely axiomatizable logic, its finite model property implies its decidability (Harrop's theorem).

Theorem 5. *When $n > m \geqslant 1$ and $h \geqslant 1$, the logics of classes $\mathcal{F}(m, n, h)$ and $\mathcal{G}(m, h)$ are decidable.*

Proof. The logics of these classes are defined by a finite set of axioms, and have the finite model property by Theorem 2. □

§ 6. Some corollaries

The finiteness of a frame is equivalent to the condition that its height, all levels of its skeleton, and the number of points in each of its clusters are finite. Under the condition that a given pretransitive (m, n)-frame has finite height, Theorem 1 and Lemma 13 enable one to bound the other parameters.

Corollary 33. *Let $n > m \geqslant 1$. Then*

1) $\operatorname{Log}\mathcal{F}(m, n)$ *has the finite model property if and only if it coincides with* $\operatorname{Log}\mathcal{F}_*(m, n)$;
2) $\operatorname{Log}\mathcal{G}(m)$ *has the finite model property if and only if it coincides with* $\operatorname{Log}\mathcal{G}_*(m)$.

What properties of the logics $\operatorname{Log}\mathcal{F}_*(m, n)$ and $\operatorname{Log}\mathcal{G}_*(m)$ can be found, along with the established finite model property? In particular, do they have a finite axiomatization, and are they decidable?

The finite model property can also imply the decidability of a modal logic without establishing its complete finite axiomatization: $\mathrm{Log}\,\mathcal{F}$ is decidable if there is a computable function f such that every formula φ satisfiable in $\mathcal{F}$ is satisfiable in a frame in $\mathcal{F}$ whose size does not exceed $f(l(\varphi))$ and the membership problem for the class $\mathcal{F}$ is decidable for any finite frame. The latter holds when, for example, the class of frames is given by finitely many modal formulae or first-order sentences. The condition on the size of the frame for the classes admitting f-bounded minimal filtrations is given by Proposition 5. We shall find these conditions for the classes $\mathcal{F}(m,n,h)$ and $\mathcal{G}(m,h)$, $n > m \geqslant 0$, $h \geqslant 1$.

Proposition 34. *Let $F = (W, R)$ be a frame, let $\mathcal{A}$ be a partition of W, let $\mathcal{B}$ be a partition of $\mathcal{A}$ and let $\mathcal{C} = \{\cup B \mid B \in \mathcal{B}\}$. Then the frames $(F_{\mathcal{A}})_{\mathcal{B}}$ and $F_{\mathcal{C}}$ are isomorphic.*

Proof. Obviously, $\mathcal{C}$ is a partition of W, and the map $B \mapsto \cup B$ is a bijection between $\mathcal{B}$ and $\mathcal{C}$.

By definition, $F_{\mathcal{A}} = (\mathcal{A}, R_{\mathcal{A}})$, $(F_{\mathcal{A}})_{\mathcal{B}} = (\mathcal{B}, (R_{\mathcal{A}})_{\mathcal{B}})$ and $F_{\mathcal{C}} = (\mathcal{C}, R_{\mathcal{C}})$. Let us verify that for every $B_1, B_2 \in \mathcal{B}$ we have $B_1 (R_{\mathcal{A}})_{\mathcal{B}} B_2 \;\Leftrightarrow\; (\cup B_1) R_{\mathcal{C}} (\cup B_2)$:

$$
\begin{aligned}
B_1 (R_{\mathcal{A}})_{\mathcal{B}}\, B_2 \quad &\Leftrightarrow \quad \exists\, A_1 \in B_1 \;\; \exists\, A_2 \in B_2 \;\; A_1 R_{\mathcal{A}} A_2 \\
&\Leftrightarrow \quad \exists\, A_1 \in B_1 \;\; \exists\, A_2 \in B_2 \;\; (\exists\, x_1 \in A_1 \;\; \exists\, x_2 \in A_2 \;\; x_1 R x_2) \\
&\Leftrightarrow \quad \exists\, x_1 \in \cup B_1 \;\; \exists\, x_2 \in \cup B_2 \;\; x_1 R x_2 \\
&\Leftrightarrow \quad (\cup B_1)\, R_{\mathcal{C}}\, (\cup B_2).
\end{aligned}
$$

This completes the proof of the proposition. □

Theorem 6. *The classes $\mathcal{F}(m,n,h)$, $\mathcal{F}_*(m,n)$, $\mathcal{G}(m,h)$ and $\mathcal{G}_*(m)$ admit minimal filtrations for all $n > m \geqslant 1$ and $h \geqslant 1$. Moreover, there are polynomials $p(x,y,z)$ and $q(x,y)$ such that, for all $n > m \geqslant 1$ and $h \geqslant 1$, the class $\mathcal{F}(m,n,h)$ admits f-bounded minimal filtrations with*

$$f(x) = \exp_2^h(p(x,h,n-m)),$$

and the class $\mathcal{G}(m,h)$ admits f-bounded minimal filtrations with

$$f(x) = \exp_2^h(q(x,h)).$$

Proof. Let $F \in \mathcal{F}(m,n,h)$, let $\mathcal{A}_0$ be a finite partition of the frame F, and let $\mathcal{A}$ be a refinement of $\mathcal{A}_0$ such that $F_{\mathcal{A}} \in \mathcal{F}(m,n,h)$ and the cardinality of the clusters in $F_{\mathcal{A}}$ are bounded by $(n-m)|\mathcal{A}_0|$ (Lemma 13). Consider the partition

$$\mathcal{B}_0 = \bigl\{\{U \in \mathcal{A} \mid U \subseteq V\} \mid V \in \mathcal{A}_0\bigr\}.$$

Obviously, $|\mathcal{A}_0| = |\mathcal{B}_0|$. Let $\mathcal{B}$ be a tuned finite refinement of $\mathcal{B}_0$ in the frame $F_{\mathcal{A}}$, which exists by Theorem 1. By Propositions 7 and 21 we have $(F_{\mathcal{A}})_{\mathcal{B}} \in \mathcal{F}(m,n,h)$. By Proposition 34, the frame thus constructed is isomorphic to $F_{\mathcal{C}}$ with $\mathcal{C} = \{\cup B \mid B \in \mathcal{B}\}$. It remains to note that $\mathcal{C}$ is a refinement of $\mathcal{A}_0$.

If x is the size of the original finite partition $\mathcal{A}_0$, then all the clusters in $F_{\mathcal{A}}$ are bounded by $(n-m)x$, and the size of the frame $(F_{\mathcal{A}})_{\mathcal{B}}$ thus constructed is bounded by

$$\exp_2^h\big(((n-m)x+h+1)((n-m)x+\log_2 x)\big).$$

The argument for the classes $\mathcal{G}(m,h)$ and $\mathcal{G}_*(m)$ is carried out in a similar way but using Lemma 18 instead of Lemma 13. The size of the frame thus constructed is bounded by

$$\exp_2^h\big((x+h+1)(x+\log_2 x)\big).$$

This completes the proof of the theorem. □

This theorem implies the decidability of the logics of classes $\mathcal{F}(m,n,h)$ and $\mathcal{G}(m,h)$ for given $n > m \geqslant 1$ and $h \geqslant 1$ in Kalmár elementary time (the decision algorithm is working in time bounded by a tower of exponentials).

Question 3. *What is the complexity of the logics of classes $\mathcal{F}(m,n,h)$ and $\mathcal{G}(m,h)$ for $n > m \geqslant 2$ and $h \geqslant 1$?*

The theorem proved above gives no effective bound for the size of filtrations in the classes $\mathcal{F}_*(m,n)$ and $\mathcal{G}_*(m)$, and the decidability problem for these logics is open for $n > m \geqslant 2$. The axiomatics of these classes is also unknown.

Question 4. *Are $\operatorname{Log}\mathcal{F}_*(m,n)$ and $\operatorname{Log}\mathcal{G}_*(m)$ decidable for $n > m \geqslant 2$?*

Question 5. *Are $\operatorname{Log}\mathcal{F}_*(m,n)$ and $\operatorname{Log}\mathcal{G}_*(m)$ finitely axiomatizable for $n > m \geqslant 2$?*

By Corollary 33, a negative answer to one of these questions will mean the absence of the finite model property of the logics of classes $\mathcal{F}(m,n)$ or $\mathcal{G}(m)$.

It seems to us quite possible that for $n > m \geqslant 2$ the logics of the classes $\mathcal{F}_*(m,n)$ and $\mathcal{G}_*(m)$ are at least not Kalmár elementary.

Let us summarize the known facts and open questions concerning the finite model property, decidability, and finite axiomatizability for $n > m > 1$ and $h \geqslant 1$:

	$\mathcal{F}(m,n,h)$, $\mathcal{G}(m,h)$	$\mathcal{F}_*(m,n)$, $\mathcal{G}_*(m)$	$\mathcal{F}(m,n)$, $\mathcal{G}(m)$
FMP	+	+	?
Decidability	+	?	?
FAx	+	?	+

We now give another consequence of the results obtained above.

The formulae of the language with k modalities are interpreted in structures with k relations:

$$M,w \models \Diamond_i\varphi \qquad \Leftrightarrow \qquad \exists\, v(wR_iv \text{ and } M,v \models \varphi).$$

Consider the language with two modalities. For a class $\mathcal{F}$ consisting of frames with one relation we set

$$\mathcal{F}^{\pm} = \{(W,R,R^{-1}) \mid (W,R) \in \mathcal{F}\}.$$

By recent results (cf. Theorem 2.7 of [28]), the filterability of a class $\mathcal{F}$ of frames implies the finite model property not only of $\operatorname{Log}\mathcal{F}$ but also of $\operatorname{Log}\mathcal{F}^{\pm}$.

Corollary 35. *Let $n > m \geqslant 1$ and $h \geqslant 1$. The logics of the classes*

$$\mathcal{F}(m,n,h)^{\pm},\ \mathcal{F}_{*}(m,n)^{\pm},\ \mathcal{G}(m,h)^{\pm},\ \mathcal{G}_{*}(m)^{\pm}$$

have the finite model property.

It is well known that if L is a canonical logic and $\mathcal{F}$ is the class of all frames of L, then the logic $\operatorname{Log}\mathcal{F}^{\pm}$ is obtained from L by adding the axioms $p \to \Box_1\Diamond_2 p$ and $p \to \Box_2\Diamond_1 p$.

Corollary 36. *For $n > m \geqslant 1$ and $h \geqslant 1$ the logics of the classes $\mathcal{F}(m,n,h)^{\pm}$ and $\mathcal{G}(m,h)^{\pm}$ are decidable.*

Despite the theorems on the finite model property proved above, the logics of pretransitive frames of finite height are significantly more complicated than the corresponding extensions of the logic S4 of preorders. For example, all the logics $\mathrm{S4}+\{B_h\}$ are locally tabular, and the logic $\mathrm{S4}+\{B_1\}=\mathrm{S5}$ is pretabular. However, neither local tabularity nor pretabularity hold even for $\operatorname{Log}\mathcal{G}(2,1)$ because this logic has Kripke incomplete extensions ([29], [30]). The same holds for all the logics $\operatorname{Log}\mathcal{F}(m,n,h)$ and $\operatorname{Log}\mathcal{G}(m,h)$, $n > m \geqslant 2$, $h \geqslant 1$, because they are included in $\operatorname{Log}\mathcal{G}(2,1)$.

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Earlier, in [24], the authors considered logics of pretransitive frames in which the relation R^* is universal (‘pretransitive analogues of S5’), that is, in fact, logics of pretransitive frames of height 1. The idea of considering the case of logics of arbitrary finite height was suggested to the authors by V. B. Shehtman, and the authors express their gratitude to him for this suggestion.

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