Locally tabular polymodal logics

Ilya B. Shapirovsky

Institute for Information Transmission Problems of the Russian Academy of Sciences
shapir@iitp.ru

A logic L is *locally tabular* if, for any finite l, there exist only finitely many pairwise nonequivalent formulas in L built from the variables $p_1, ..., p_l$. Equivalently, a logic L is locally tabular if the variety of its algebras is locally finite, i.e., every finitely generated L-algebra is finite. This is a very strong property: if a logic is locally tabular, then it has the finite model property (thus it is Kripke complete); every extension of a locally tabular logic is locally tabular (thus it has the finite model property); every finitely axiomatizable extension of a locally tabular logic is decidable.

According to the classical results by Segerberg and Maksimova [4, 3], a unimodal logic containing K4 is locally tabular iff it is of finite height. The notion of finite height can also be defined for logics, in which the master modality is expressible ('pretransitive' logics). Recently [5], it was shown that every locally tabular unimodal logic is a pretransitive logic of finite height. Also, in [5], two semantic criteria of local tabularly for unimodal logics were proved. In this note we formulate the analogs of these facts for the polymodal case and discuss some of their corollaries.

Fix some $n > 0$ and the n-modal language with the modalities $\Diamond_0, \ldots, \Diamond_{n-1}$.

Necessary condition. For a Kripke frame $\mathrm{F} = (W, (R_i)_{i<n})$, put $R_\mathrm{F} = \cup_{i<n} R_i$. Let $\sim_\mathrm{F}$ be the equivalence relation $R^*_\mathrm{F} \cap {R^*_\mathrm{F}}^{-1}$, where R^*_F denotes the transitive reflexive closure of R_F. A *cluster* in F is an equivalence class under $\sim_\mathrm{F}$. For clusters $C,\ D$, put $C \leq_\mathrm{F} D$ iff $x R^*_\mathrm{F} y$ for some $x \in C, y \in D$. The poset $(W/{\sim_\mathrm{F}}, \leq_\mathrm{F})$ is called the *skeleton of* F.

A poset is of *finite height* $\leq h$ if every of its chains contains at most h elements. The *height of a frame* F, in symbols, $ht(\mathrm{F})$, is the height of its skeleton.

For a binary relation R put $R^{\leq m} = \cup_{i \leq m} R^i$. R is called *m-transitive*, if $R^{m+1} \subseteq R^{\leq m}$. A frame F is *$m$-transitive* if R_F is m-transitive.

Put $\Diamond^0 \varphi = \varphi,\ \Diamond^{i+1}\varphi = \Diamond^i(\Diamond_0\varphi \vee \ldots \vee \Diamond_{n-1}\varphi),\ \Diamond^{\leq m}\varphi = \vee_{i\leq m}\Diamond^i\varphi,\ \Box^{\leq m}\varphi = \neg\Diamond^{\leq m}\neg\varphi$.

Proposition 1. *A frame* F *is m-transitive iff* $\mathrm{F} \vDash \Diamond^{m+1}p \to \Diamond^{\leq m}p$.

Put $B_1^{[m]} = p_1 \to \Box^{\leq m}\Diamond^{\leq m}p_1,\ B_{i+1}^{[m]} = p_{i+1} \to \Box^{\leq m}(\Diamond^{\leq m}p_{i+1} \vee B_i^{[m]})$.

Proposition 2. *For an m-transitive frame* F, $\mathrm{F} \vDash B_h^{[m]}$ *iff* $ht(\mathrm{F}) \leq h$.

A logic L is called *m-transitive* if $L \vdash \Diamond^{m+1}p \to \Diamond^{\leq m}p$. L is *pretransitive* if it is m-transitive for some $m \geq 0$. An m-transitive logic L is of *finite height* $\leq h$ if $L \vdash B_h^{[m]}$.

Theorem 1. *Every locally tabular logic is a pretransitive logic of finite height.*

Being a pretransitive logic of finite height is not sufficient for local tabularity. For example, products of transitive logics are pretransitive, in particular, the logic $\mathrm{S5} \times \mathrm{S5}$ is a pretransitive logic of height 1. It is known to be not locally tabular (still, it is pre-locally tabular [1]).

First criterion. As usual, *a partition* $\mathcal{A}$ of a set W is a set of non-empty pairwise disjoint sets such that $W = \cup\mathcal{A}$. A partition $\mathcal{B}$ *refines* $\mathcal{A}$, if each element of $\mathcal{A}$ is the union of some elements of $\mathcal{B}$.

Definition 1. Let $\mathrm{F} = (W, (R_i)_{i<n})$ be a Kripke frame. A partition $\mathcal{A}$ of W is F-*tuned*, if for every $U, V \in \mathcal{A}$, and every $i < n$

$$\exists u \in U\ \exists v \in V\ uR_iv \quad \Rightarrow \quad \forall u \in U\ \exists v \in V\ uR_iv.$$

A class of frames $\mathcal{F}$ is *ripe*, if there exists $f : \mathbb{N} \to \mathbb{N}$ such that for every $\mathrm{F} \in \mathcal{F}$ and every finite partition $\mathcal{A}$ of F there exists an F-tuned refinement $\mathcal{B}$ of $\mathcal{A}$ with $|\mathcal{B}| \leq f(|\mathcal{A}|)$.

Theorem 2. *A logic L is locally tabular iff L is the logic of a ripe class of frames.*

Let L_u be the extension of L with the universal modality. Trivially, if a partition is tuned for a frame $(W, (R_i)_{i<n})$, then it is tuned for the frame $(W, (R_i)_{i<n}, W \times W)$.

Corollary 1. *If L is locally tabular, then L_u is locally tabular.*

Let L_t be the tense counterpart of L. From Theorem 2 and the filtration technique proposed in [2, Theorem 2.4], we have

Corollary 2. *If L is locally tabular, then L_t has the finite model property.*

Second criterion. For a class $\mathcal{F}$ of frames let $cl\mathcal{F}$ be the class of restrictions on clusters occurring in frames from $\mathcal{F}$: $cl\mathcal{F} = \{\mathrm{F}{\restriction}C \mid \mathrm{F} \in \mathcal{F} \text{ and } C \text{ is a cluster in F}\}$. $\mathcal{F}$ has the *ripe cluster property* if $cl\mathcal{F}$ is ripe. A class $\mathcal{F}$ of frames is of *finite height* if there exists $h \in \mathbb{N}$ such that $ht(\mathrm{F}) \leq h$ for all $\mathrm{F} \in \mathcal{F}$.

Theorem 3. *$\mathcal{F}$ is ripe iff $\mathcal{F}$ is of finite height and has the ripe cluster property.*

A logic has the *ripe cluster property* if the class of all its frames has. Note that if a pretransitive logic is canonical, then its extensions with formulas of finite height are canonical, thus are Kripke complete. From Theorems 2 and 3, we obtain

Corollary 3. *Suppose L_0 is a canonical pretransitive logic with the ripe cluster property. Then for any logic $L \supseteq L_0$, L is locally tabular iff L is of finite height.*

It is known that a unimodal logic $L \supseteq \mathrm{K4}$ is locally tabular iff its 1-generated free algebra $\mathfrak{A}_L(1)$ is finite [3]. It allows us to formulate another corollary of Theorem 3:

Corollary 4. *Suppose L_0 is a canonical pretransitive logic with the ripe cluster property. Then for any logic $L \supseteq L_0$, L is locally tabular iff $\mathfrak{A}_L(1)$ is finite.*

Problem. *Does this equivalence hold for every modal logic?*

References

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