ISSN 0032-9460, Problems of Information Transmission, 2007, Vol. 43, No. 3, pp. 255–262. © Pleiades Publishing, Inc., 2007.
Original Russian Text © I.B. Shapirovsky, 2007, published in Problemy Peredachi Informatsii, 2007, Vol. 43, No. 3, pp. 97–104.

LARGE SYSTEMS

Modal Logics of Some Geometrical Structures[1]

I. B. Shapirovsky

Kharkevich Institute for Information Transmission Problems, RAS, Moscow
shapir@iitp.ru

Received June 19, 2007

Abstract—We study modal logics of regions in a real space ordered by the inclusion and compact inclusion relations. For various systems of regions, we propose complete finite modal axiomatizations; the described logics are finitely approximable and PSPACE-complete.

DOI: 10.1134/S0032946007030076

1. INTRODUCTION

One of intensively developing fields in theoretical computer science is the study of the so-called *spatial* logics, used for the description of geometrical and topological relations and operations, and in particular, investigation of semantic and algorithmic properties of these logics. This development is due to, one the one hand, applied problems (such as pattern recognition or design of geoinformation systems) and on the other hand, issues of axiomatization of various areas of mathematics and mathematical physics (for instance, topology and theory of relativity).

In particular, logical calculi that axiomatize relations between *regions* (i.e., sets of certain type in a topological space) are actively being investigated nowadays. For these purposes, various systems have been proposed in classical first-order languages, as well as in languages of modal logic.

As examples of relations between regions, one may consider $\subseteq$ (*inclusion*), $\Subset$ (*compact inclusion*: $A \Subset B \Leftrightarrow \mathbf{C}A \subseteq \mathbf{I}B$, where $\mathbf{I}$ and $\mathbf{C}$ are the interior and the closure operators, respectively), their converse $\supseteq$ and $\Supset$, $\ominus$ (*disconnectedness*: $A \ominus B \Leftrightarrow A \cap B = \varnothing$), $\between$ (*partial overlapping*: $A \between B \Leftrightarrow \mathbf{I}A \cap \mathbf{I}B \neq \varnothing \wedge A \nsubseteq B \wedge B \nsubseteq A$), etc.

If we consider a set of regions with relations as a Kripke-frame, then the question on its modal logic arises. Recently, it was shown in [1] that modal logics of regions with several relations can be undecidable or even not recursively enumerable (in [1], among others, modal logics of regular sets ordered by the eight Egenhofer-Franzosa relations RCC-8 [2,3] were considered). In [4] it was shown that there is a number of monomodal logics of regions which are not finitely axiomatizable. This leads to the problem to find expressive modal systems with "good" properties (finite axiomatizability, finite approximability, appropriate computational complexity).

In this paper we consider frames of the form $(\mathcal{W}, R)$, where R is one of the relations $\subseteq$, $\Subset$, $\supseteq$, or $\Supset$ and $\mathcal{W}$ is a nonempty set of *n-regions*; by an *n-region* we mean the closure of a domain in $\mathbb{R}^n$. Logics of these frames were described in [4] for the cases where $\mathcal{W}$ consists of all (convex) n-regions, n-dimensional balls, or bricks. In this paper we generalize this result: we define a special class of *saturated* sets of regions and describe the logics of $(\mathcal{W}, R)$ for any saturated $\mathcal{W}$; in particular, any nonempty set of convex regions closed under homotheties is saturated (Theorems 2 and 3). We also axiomatize logics of some bimodal frames with the additional universal relation $\mathcal{W}\times\mathcal{W}$ (Theorem 5). All described logics are finitely axiomatizable, finitely approximable, and PSPACE-complete.

[1] Supported in part by the Russian Foundation for Basic Research, project no. 06-01-72555, and RFBR-NWO, project no. 047.011.2004.04.

2. PRELIMINARIES

Let us recall some basic notions from the theory of modal logic.

Definition 1. The set of n-modal formulas $FM(\Diamond_1, \ldots, \Diamond_n)$ is defined via a countable set of propositional variables $PV = \{p_1, p_2, \ldots\}$, propositional constant $\perp$ (*false*), binary connective $\to$, and unary connectives $\Diamond_1, \ldots, \Diamond_n$ (*possibility operators*) as follows:

- $\perp, p_1, p_2, \ldots$ are formulas;
- If A, B are formulas, then $(A \to B)$ is a formula;
- If A is a formula, then $\Diamond_1 A, \ldots, \Diamond_n A$ are formulas.

The connectives $\neg$, $\wedge$, $\vee$, and $\top$ are defined in the standard way; in particular, $\neg A := A \to \perp$ and $\top := \perp \to \perp$. Also, we put

$$\Box_i A := \neg \Diamond_i \neg A \quad (\textit{necessity operators}).$$

Definition 2. A (*normal*) *n-modal logic* is a set of n-modal formulas $\mathrm{L} \subseteq FM(\Diamond_1, \ldots, \Diamond_n)$ such that

- L contains all classical tautologies,
- L contains the formulas $\neg \Diamond_i \perp$, $i = 1, \ldots, n$,
- L contains the formulas $\Diamond_i(p_1 \vee p_2) \to \Diamond_i p_1 \vee \Diamond_i p_2$, $i = 1, \ldots, n$,
- L is closed under the *modus ponens rule*, *uniform substitution rule*, and *monotonicity rules* Mon_i, $i = 1, \ldots, n$:

$$\mathrm{Mon}_i: \qquad \text{if} \quad A \to B \in \mathrm{L}, \quad \text{then} \quad \Diamond_i A \to \Diamond_i B \in \mathrm{L}.$$

Definition 3. An *n-modal Kripke frame* (or simply an *n-frame*) F is a tuple $(W, R_1, \ldots, R_n)$, where W is a nonempty set and $R_1, \ldots, R_n \subseteq W \times W$. A *valuation* on F is a map

$$\theta\colon PV \to \mathcal{P}(W),$$

where $\mathcal{P}(W)$ is the power-set of W. A *Kripke model* M over a frame F is a pair (F, θ), where θ is a valuation on F.

The trues of a modal formula A in a model M at a point x (notation: $M, x \vDash A$) is defined as follows:

$$\begin{array}{lll} M, x \nvDash \perp, & & \\ M, x \vDash p & \Leftrightarrow & x \in \theta(p), \\ M, x \vDash A \to B & \Leftrightarrow & M, x \nvDash A \text{ or } M, x \vDash B, \\ M, x \vDash \Diamond_i A & \Leftrightarrow & \text{for some } y \text{ we have } xR_i y \text{ and } M, y \vDash A. \end{array}$$

A formula A is *true in a model M* if it is true at all points in M; A is *valid in a frame F* (notation: $F \vDash A$) if it is true in any model over F; A is valid *in a class $\mathcal{F}$ of frames* (notation: $\mathcal{F} \vDash A$) if A is valid in any frame $F \in \mathcal{F}$. The set of all valid formulas in a class $\mathcal{F}$ is denoted by $\mathrm{L}(\mathcal{F})$. The notation $\mathrm{L}(F)$ abbreviates $\mathrm{L}(\{F\})$.

Proposition 1 [5]. *For a class $\mathcal{F}$, the set of formulas $\mathrm{L}(\mathcal{F})$ is a modal logic.*

Definition 4. A logic L is *complete with respect to a class $\mathcal{F}$ of frames* if $\mathrm{L} = \mathrm{L}(\mathcal{F})$. A logic L is called (*Kripke-*) *complete* if it is complete with respect to some class of frames. A logic L is called finitely approximable if it is complete with respect to some class of finite frames.

Definition 5. Consider frames $F = (W, R)$ and $G = (V, S)$; let f be a map from W to V. We say that f is *monotonic* if

$$\forall x \in W \; \forall y \in W \; (xRy \to f(x)Sf(y));$$

we say that f has the *lift property* if

$$\forall x \in W\ \forall z \in V\ (f(x)Sz \to \exists y\ (xRy \wedge f(y) = z)).$$

If f is surjective, monotonic, and has the lift property, then f is a p-*morphism from* F *to* G (notation $f\colon F \twoheadrightarrow G$); f is a p-*morphism from an* n*-frame* $(W, R_1, \ldots, R_n)$ *to an* n*-frame* $(V, S_1, \ldots, S_n)$ if $f\colon (W, R_i) \twoheadrightarrow (V, S_i)$ for all $i = 1, \ldots, n$. Finally, $F \twoheadrightarrow G$ means that there exists a p-morphism from F to G.

The following fact is well known (see, e.g., [5]).

Lemma 1 (p-morphism lemma). *If* $F \twoheadrightarrow G$, *then* $\mathrm{L}(F) \subseteq \mathrm{L}(G)$.

Consider the following modal formulas:

$$\begin{array}{ll} \mathrm{ATr} := \Diamond\Diamond p \to \Diamond p, & \mathrm{ARefl} := p \to \Diamond p, \\ \mathrm{ASer} := \Diamond\top, & \mathrm{AConf} := \Diamond\Box p \to \Box\Diamond p, \\ \mathrm{AM} := \Box\Diamond p \to \Diamond\Box p, & \mathrm{AM}' := \Box\bot \vee \Diamond\Box\bot, \\ \mathrm{ADens}_2 := \Diamond p_1 \wedge \Diamond p_2 \to \Diamond(\Diamond p_1 \wedge \Diamond p_2). & \end{array}$$

Proposition 2 [4,5]. *For a monomodal frame* $F = (W, R)$,

- $F \vDash \mathrm{ATr} \iff F$ *is transitive*: $\forall x\ \forall y\ \forall z\ (xRy \wedge yRz \to xRz)$;
- $F \vDash \mathrm{ARefl} \iff F$ *is reflexive*: $\forall x\ (xRx)$;
- $F \vDash \mathrm{AConf} \iff F$ *is* (*weakly*) *directed*: $\forall x\ \forall y_1\ \forall y_2\ \exists z\ (xRy_1 \wedge xRy_2 \to y_1Rz \wedge y_2Rz)$;
- $F \vDash \mathrm{ASer} \iff F$ *is serial*: $\forall x\ \exists y\ (xRy)$;
- $F \vDash \mathrm{ADens}_2 \iff F$ *is 2-dense:* $\forall x\ \forall y_1\ \forall y_2\ \exists z\ (xRy_1 \wedge xRy_2 \Rightarrow xRz \wedge zRy_1 \wedge zRy_2)$;
- $F \vDash \mathrm{ATr}$, $F \vDash \mathrm{AM} \iff F$ *is transitive and has the* M*-property*: $\forall x\ \exists y\ (xRy \wedge \forall z\ (yRz \to y = z))$;
- $F \vDash \mathrm{AM}' \iff F$ *has the* M$'$*-property:* $\forall x\ \exists y\ ((xRy \vee x = y) \wedge \forall z\ \neg(yRz))$.

3. MODAL LOGICS OF REGIONS

3.1. Regions in $\mathbb{R}^n$

For a set V in a topological space, let $\mathbf{I}V$ and $\mathbf{C}V$ denote the interior and the closure of V, respectively. Recall that if V is open and connected, then V is called a *domain*.

Definition 6. A compact set $U \subseteq \mathbb{R}^n$ is called a *region in* $\mathbb{R}^n$ (n-region, for short) if U is the closure of some nonempty domain.

For a set of n-regions $\mathcal{W}$, let $\mathcal{W}^\circ$ denote the extension of $\mathcal{W}$ by all singletons: $\mathcal{W}^\circ = \mathcal{W} \cup \{\{X\} \mid X \in \mathbb{R}^n\}$.

Let $d(X, Y)$ denote the Euclidean distance between points $X, Y \in \mathbb{R}^n$.

For $X \in \mathbb{R}^n$ and $r \geq 0$, let $B(X, r)$ denote the closed ball of radius r centered at X; for $U \subseteq \mathbb{R}^n$, let

$$B(U, r) := \{X \mid \text{ for some } Y \in U\ d(X, Y)\ \leq r\}.$$

Consider the following sets of regions in $\mathbb{R}^n$:

$\mathcal{R}eg_n$, the set of all n-regions;
$\mathcal{C}onv_n$, the set of all convex n-regions;
$\mathcal{B}_n = \{B(X, r) \mid X \in \mathbb{R}^n,\ r > 0\}$, the set of all balls;
$\mathcal{R}_n = \Big\{ \prod\limits_{i=1}^{n} [a_i, b_i] \mid [a_1, b_1], \ldots, [a_n, b_n] \in \mathcal{B}_1 \Big\}$, the set of all n-*bricks*.

Observe that if $n = 1$, then $\mathcal{R}eg_1 = \mathcal{C}onv_1 = \mathcal{R}_1 = \mathcal{B}_1$ and all these sets coincide with the set of all segments on the real line.

For sets $U, V \subseteq \mathbb{R}^n$, put

$$U \Subset V := \mathbf{C}U \subseteq \mathbf{I}V, \qquad \Supset := \Subset^{-1}.$$

We study the logics of regions ordered by the relations $\subseteq$ and $\Subset$ (and their converse $\supseteq$ and $\Supset$).

Let K be the minimal modal logic; for an axiom A and a logic L, let L+A be the minimal modal logic that contains L and A. Put

$$\begin{array}{ll} \mathrm{K4} = \mathrm{K} + \mathrm{ATr}, & \mathrm{S4} = \mathrm{K4} + \mathrm{ARefl}, \\ \mathrm{S4.1} = \mathrm{K4} + \mathrm{AM}, & \mathrm{S4.2} = \mathrm{S4} + \mathrm{AConf}, \\ \mathrm{LM}_0 = \mathrm{K4} + \mathrm{ADens}_2, & \mathrm{LM}_1 = \mathrm{LM}_0 + \mathrm{ASer}, \\ \mathrm{LM}_2 = \mathrm{LM}_1 + \mathrm{AConf}, & \mathrm{LM}_3 = \mathrm{LM}_0 + \mathrm{AM}'. \end{array}$$

Theorem 1 (see [4]). *Let $\mathcal{W} \in \{\mathcal{R}eg_n, \mathcal{C}onv_n, \mathcal{R}_n, \mathcal{B}_n\}$ and $R \in \{\subseteq, \supseteq, \Subset, \Supset\}$. Then the logics of frames $(\mathcal{W}, R)$ and $(\mathcal{W}^\circ, R)$ are described in the following table*:

The logics of frames $(\mathcal{W}, R)$ and $(\mathcal{W}^\circ, R)$, $R \in \{\subseteq, \supseteq, \Subset, \Supset\}$

	$\Supset$	$\supseteq$	$\Subset$	$\subseteq$
$\mathcal{W}$	LM_1	S4	LM_2	S4.2
$\mathcal{W}^\circ$	LM_3	S4.1	LM_2	S4.2

3.2. Saturated Sets of Regions

For a function $f\colon A \to B$, let $f_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$, where $f_*(C) = \{f(x) \mid x \in C\}$ for any $C \subseteq A$.

For a set $V \subseteq \mathbb{R}^n$, let $-V$ denote the complement of V in $\mathbb{R}^n$.

Definition 7. A set of n-regions $\mathcal{W}$ is called *saturated* if

$$\forall u \in \mathcal{B}_n\ \exists v \in \mathcal{W}\ \ (u \subseteq v)\,, \tag{1}$$

$$\forall u \in \mathcal{W}\ \forall \varepsilon > 0\ \exists v \in \mathcal{W}\ \ (u \Subset v \subseteq B(u, \varepsilon))\,, \tag{2}$$

$$\forall u \in \mathcal{W}\ \forall \varepsilon > 0\ \exists v \in \mathcal{W}\ \ (-B(-u, \varepsilon) \subseteq v \Subset u)\,, \tag{3}$$

and there exists an open continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that for $R \in \{\subseteq, \supseteq, \Subset, \Supset\}$,

$$\forall u \in \mathcal{W}^\circ\ \forall s \in \mathcal{R}eg_1^\circ\ \ (f_*(u) R s \to \exists w \in \mathcal{W}^\circ\ \ (uRw \wedge f_*(w) = s))\,. \tag{4}$$

Property (4) corresponds to the lift property of the map f_* from the frame $(\mathcal{W}^\circ, R)$ to the frame $(\mathcal{R}eg_1^\circ, R)$.

Proposition 3. *Let $\mathcal{W}$ be a saturated set of n-regions, $n \geq 1$, $R \in \{\subseteq, \supseteq, \Subset, \Supset\}$. Then*

$$f_*\colon (\mathcal{W}, R) \twoheadrightarrow (\mathcal{R}eg_1, R), \qquad f_*\colon (\mathcal{W}^\circ, R) \twoheadrightarrow (\mathcal{R}eg_1^\circ, R).$$

Proof. Since f is continuous, the image of any $u \in \mathcal{W}^\circ$ is a segment or a point. If $u \in \mathcal{W}$, then $\mathbf{I}u \neq \varnothing$, and since f is open, $\mathbf{I}f_*(u) \neq \varnothing$; i.e., $f_*(u)$ is a segment. Therefore,

$$f_*\colon \mathcal{W} \to \mathcal{R}eg_1, \qquad f_*\colon \mathcal{W}^\circ \to \mathcal{R}eg_1^\circ. \tag{5}$$

Let us check the surjectivity. Let $s \in \mathcal{W}^\circ$. Due to (1), $\mathcal{W} \neq \varnothing$. Let $v_0 \in \mathcal{W}$, $f_*(w) = s_0 \in \mathcal{R}eg_1^\circ$. Consider the segment s_1 that contains $s_0 \cup s$. Due to (4), $s_0 \subseteq s_1$ implies $f_*(v_1) = s_1$ for some

$v_1 \in \mathcal{W}^\circ$; similarly, $s_1 \supseteq s$ implies $f_*(v) = s$ for some $v \in \mathcal{W}^\circ$. Thus, the map $f_*\colon \mathcal{W}^\circ \to \mathcal{R}eg_1^\circ$ is surjective, and due to (5), $f_*\colon \mathcal{W} \to \mathcal{R}eg_1$ is surjective.

Let us show that f in monotonic. For $u, v \in \mathcal{W}^\circ$, if $u \subseteq v$, then $f_*(u) \subseteq f_*(v)$. Let $u \Subset v$. Then $u \subseteq \mathbf{I}v \subseteq v$, and $f_*(u) \subseteq f_*(\mathbf{I}v) \subseteq f_*(v)$. The set $f_*(\mathbf{I}v)$ is open because f is open; thus, $f_*(u) \Subset f_*(v)$.

The lift property immediately follows from (4).

One can show that the sets of regions $\mathcal{R}eg_n, \mathcal{C}onv_n, \mathcal{R}_n$, and $\mathcal{B}_n$ are saturated: properties (1)–(3) are checked by straightforward arguments, and to show (4), it suffices to consider the projection $\mathsf{Pr}_1\colon (x_1, \ldots, x_n) \mapsto x_1$ (see [4] for more details).

The following theorem is a generalization of Theorem 1.

Theorem 2. *The table of Theorem* 1 *describes the logics of any saturated set of n-regions, $n \geq 1$.*

Proof. From the p-morphism lemma and Proposition 3, we have

$$\mathrm{L}(\mathcal{W}, R) \subseteq \mathrm{L}(\mathcal{R}eg_1, R), \qquad \mathrm{L}(\mathcal{W}^\circ, R) \subseteq \mathrm{L}(\mathcal{R}eg_1^\circ, R).$$

Let us check the converse inclusions. Due to Proposition 2 and Theorem 1, it suffices to check the corresponding first-order properties of the relations $\subseteq$, $\supseteq$, $\Subset$, and $\Supset$.

All these relations are transitive; the relations $\subseteq$ and $\supseteq$ are reflexive. The frame $(\mathcal{W}^\circ, \supseteq)$ has the M-property: if $u \in \mathcal{W}^\circ$, then for $X \in u$ we have $u \supseteq \{X\}$, and if $\{X\} \supseteq v$, then $v = \{X\}$. Similarly, the frame $(\mathcal{W}^\circ, \Supset)$ has the M$'$-property. Directedness of the relations $\subseteq$ and $\Subset$ follows from (1) because any u_1 and u_2 from $\mathcal{W}^\circ$ are (compactly) contained in some ball $u \in \mathcal{B}_n$. Also, this implies seriality for the frames $(\mathcal{W}, \Subset)$ and $(\mathcal{W}^\circ, \Subset)$; seriality of $(\mathcal{W}, \Supset)$ follows from (3).

Now let us check 2-density of the relations $\Subset$ and $\Supset$. Let $u \Subset v_1$ and $u \Subset v_2$. Since u is compact, for some $\varepsilon > 0$ we have $B(u, \varepsilon) \Subset u_1 \cap u_2$. Due to (2), $u \Subset v \subseteq B(u, \varepsilon)$ for some $v \in \mathcal{W}$. Thus, $u \Subset v$, $v \Subset v_1$, and $v \Subset v_2$. Similarly, if $u \Supset v_1$ and $u \Supset v_2$, then due to (3) there exists a region $v \in \mathcal{W}$ such that $u \Supset v \Supset u_1 \cup u_2$.

Consider the following example. For a region $u \in \mathcal{R}eg_n$, let $\mathfrak{H}(u)$ be the minimal set of regions that contains u and is closed under homotheties of $\mathbb{R}^n$. In particular, $\mathcal{B}_n = \mathfrak{H}(B(X, r))$ for arbitrary $X \in \mathbb{R}^n$ and $r > 0$. Let $u = B(X, r_1) - \mathbf{I}B(X, r_2)$, $X \in \mathbb{R}^n$, $r_1 > r_2 > 0$. It is easy to see that the set of regions $\mathcal{W} = \mathfrak{H}(u)$ is not saturated. Moreover, if u is a convex region, then $\mathcal{W} = \mathfrak{H}(u)$ happens to be saturated due to the following proposition.

Proposition 4. *If $\mathcal{W}$ is a nonempty set of convex regions closed under homotheties of $\mathbb{R}^n$, then $\mathcal{W}$ is saturated.*

Proof. For $O \in \mathbb{R}^n$ and $k \neq 0$, let H_O^k denote the homothety with center 0 and coefficient k.

Let us check (1). Let $v \in \mathcal{W}$. Since $\mathbf{I}v \neq \varnothing$, we have $B(X, \varepsilon) \subseteq v$ for some $X \in v$ and $\varepsilon > 0$. Thus, for an arbitrary ball $u \in \mathcal{B}_n$ we have $u \subseteq H_X^k(B(X, \varepsilon))$ for sufficiently large k, which implies (1).

Note that if $u \in \mathcal{W}$, $k > 1$, $O \in \mathbf{I}u$, then $u \Subset H_O^k(u)$. Indeed, $O \in \mathbf{I}H_O^k(u)$, and the inverse image of any $Y \neq O$ under H_O^k belongs to the interval OY; due to the convexity, any point of the interval OY belongs to the interior of u; thus, its image belongs to the interior of $H_O^k(u)$. Therefore, $Y \in \mathbf{I}H_O^k(u)$ and $u \Subset H_O^k(u)$. Analogously, if $O \in u$, then $u \subseteq H_O^k(u)$.

Let us check properties (2) and (3). Let $u \in \mathcal{W}$ and $\varepsilon > 0$. Consider a point $O \in \mathbf{I}u$ and put $a = \sup\{d(O, X) \mid X \in u\}$. To check (2), put

$$k = 1 + \frac{\varepsilon}{a}, \qquad v = H_O^k(u).$$

Then $v \in \mathcal{W}$, $u \Subset v$. For $X \in u$, $X' = H_O^k(X)$, we have

$$d(X, X') = (k-1)d(O, X) \leq (k-1)a = \varepsilon;$$

i.e., $X' \in B(u, \varepsilon)$. Thus, $v \subseteq B(u, \varepsilon)$. Similarly, for

$$v = H_O^k(u), \quad \text{where} \quad k = 1 - \min\left(\frac{\varepsilon}{a}, \frac{1}{2}\right),$$

we have $-B(-u, \varepsilon) \subseteq v \Subset u$.

To check (4), put $f = \mathsf{Pr}_1$. Let $u \in \mathcal{W}^\circ$, $f_*(u) = [a, b]$, and $s \in \mathcal{R}eg_1^\circ = [a_1, b_1]$. Let us show that $f_*(u)Rs$ implies $uRw \wedge f(w) = s$ for some $w \in \mathcal{W}^\circ$. We consider only the case $a \neq b$, $a_1 \neq b_1$, and $R \in \{\Subset, \subseteq\}$; arguments for other cases are analogous.

Let $h(x) = (x-c)k + c$ be the transformation of $\mathbb{R}$ that maps the segment $[a, b]$ onto the segment $[a_1, b_1]$ (it is easy to find the coefficients of h: $k = \dfrac{b_1 - a_1}{b - a}$ and $c = \dfrac{ka - a_1}{k - 1}$).

Let V be the intersection of u and the hyperplane $x_1 = c$. Since $c \in [a, b]$, we have $V \neq \varnothing$. Put $w = H_O^k(u)$ for $O \in V$. Then $u \subseteq w$. Note that $\mathsf{Pr}_1(H_O^k(u)) = h(\mathsf{Pr}_1(u))$. Therefore, $f_*(w) = s$.

Moreover, if $[a, b] \Subset [a_1, b_1]$, then c is an interior point of the segment $[a, b]$, and due to the convexity of u the set V contains interior points of u. Taking one of such points as the homothety center, we obtain $u \Subset w$.

Theorem 2 and Proposition 4 immediately imply the following fact.

Theorem 3. *The table of Theorem* 1 *describes the logics of any nonempty set of convex regions closed under homotheties.*

3.3. Logics with the Universal Modality

For a monomodal frame $F = (W, R)$, let F^u be the bimodal frame with the *universal* relation:

$$F^u = (W, R, W \times W).$$

Consider the bimodal language $FM(\Diamond, \diamondsuit)$. For a monomodal logic L, let LU be the minimal bimodal logic that contains L and the formulas

$$\begin{aligned} &\diamondsuit\diamondsuit p \to \diamondsuit p, \quad p \to \diamondsuit p, \quad p \to \neg\diamondsuit\neg\diamondsuit p; \\ &\Diamond p \to \diamondsuit p. \end{aligned}$$

Then $(W, R_1, R_2) \vDash \mathrm{LU} \iff (W, R_1) \vDash \mathrm{L}$, R_2 is an equivalence relation on W, and $R_1 \subseteq R_2$ [6].

A frame $F = (W, R)$ is *downward-directed* if it satisfies the condition $\forall x\ \forall y\ \exists z\ (zRx \wedge zRy)$. Consider the following modal formula:

$$\mathrm{A}^{\downarrow} := \diamondsuit p \wedge \diamondsuit q \to \diamondsuit(\Diamond p \wedge \Diamond q).$$

For a monomodal logic L, put

$$\mathrm{LU}^{\downarrow} := \mathrm{LU} + \mathrm{A}^{\downarrow}.$$

The following proposition is straightforward.

Proposition 5 [7]. *For a monomodal frame F,*

- $F^u \vDash \mathrm{A}^{\downarrow} \iff F$ *is downward-directed*;
- *If F is downward-directed, then* $F^u \vDash \mathrm{L}(F)\mathrm{U}^{\downarrow}$.

Logics of downward-directed frames with universal modality were considered in [7]. In particular, the following theorem was proved.

Theorem 4 [7]. *For $n \geq 1$, we have*

$$\mathrm{L}((\mathcal{R}eg_n, \supseteq)^u) = \mathrm{S4U}^{\downarrow}, \qquad \mathrm{L}((\mathcal{R}eg_n^{\circ}, \supseteq)^u) = \mathrm{S4.1U}^{\downarrow};$$
$$\mathrm{L}((\mathcal{R}eg_n, \Supset)^u) = \mathrm{LM_1U}^{\downarrow}, \qquad \mathrm{L}((\mathcal{R}eg_n^{\circ}, \Supset)^u) = \mathrm{LM_3U}^{\downarrow}.$$

Let us generalize this result for the case of an arbitrary saturated set of regions.

Proposition 6. *Let F and G be monomodal frames, $F \twoheadrightarrow G$. Then $\mathrm{L}(F^u) \subseteq \mathrm{L}(G^u)$.*

Proof. Any surjective map $f\colon W \to V$ is a p-morphism of frames $(W, W \times W) \twoheadrightarrow (V, V \times V)$. Thus, $F^u \twoheadrightarrow G^u$.

Theorem 5. *Let $\mathcal{W}$ be a saturated set of n-regions, $n \geq 1$. Then*

$$\mathrm{L}((\mathcal{W}, \supseteq)^u) = \mathrm{S4U}^{\downarrow}, \qquad \mathrm{L}((\mathcal{W}^{\circ}, \supseteq)^u) = \mathrm{S4.1U}^{\downarrow};$$
$$\mathrm{L}((\mathcal{W}, \Supset)^u) = \mathrm{LM_1U}^{\downarrow}, \qquad \mathrm{L}((\mathcal{W}^{\circ}, \Supset)^u) = \mathrm{LM_3U}^{\downarrow}.$$

Proof. Let $R \in \{\supseteq, \Supset\}$. Due to Theorem 4, it suffices to show that $\mathrm{L}((\mathcal{W}, R)^u) = \mathrm{L}((\mathcal{R}eg_n, R)^u)$ and $\mathrm{L}((\mathcal{W}^{\circ}, R)^u) = \mathrm{L}((\mathcal{R}eg_n^{\circ}, R)^u)$.

Due to property (1) of saturated sets of regions, the frames $(\mathcal{W}, R)$ and $(\mathcal{W}^{\circ}, R)$ are downward-directed, and from Proposition 5 it follows that $\mathrm{L}((\mathcal{W}, R)^u) \supseteq \mathrm{L}((\mathcal{R}eg_n, R)^u)$ and $\mathrm{L}((\mathcal{W}^{\circ}, R)^u) \supseteq \mathrm{L}((\mathcal{R}eg_n^{\circ}, R)^u)$.

The converse inclusions follow from Propositions 3 and 6.

The question of axiomatization of the relations $\subseteq$ and $\Subset$ in the language with the universal modality is open.

4. CONCLUDING REMARKS

Theorems 2 and 5 give us finite modal axiomatizations for various sets of regions in $\mathbb{R}^n$. Let us consider properties of the described logics more precisely. It is well known that the logics S4, S4.1, and S4.2 are finitely approximable (see [5]); since they are finitely axiomatizable, they are decidable. Moreover, they are PSPACE-complete (see [8]). Analogous results for the logics $\mathrm{LM_1}$, $\mathrm{LM_2}$, and $\mathrm{LM_3}$ were proved recently: results on finite approximation were proved in [9], and PSPACE-completeness was proved in [10]. In [7] it was shown that these properties are preserved for the logics $\mathrm{S4U}^{\downarrow}$, $\mathrm{S4.1U}^{\downarrow}$, $\mathrm{LM_1U}^{\downarrow}$, and $\mathrm{LM_3U}^{\downarrow}$. Therefore, the logics described in Theorems 2 and 5 are finitely approximable and PSPACE-complete.

Due to Theorem 3, the logics of various sets of convex regions (balls, cylinders, cones, convex polyhedra, etc.) ordered by one of the relations $\subseteq$, $\supseteq$, $\Subset$, or $\Supset$ coincide with each other: these are the logics of the set of segments $\mathcal{R}eg_1$. However, for other natural relations between regions, the situation might be different. To illustrate this, consider the relation $\subset$: $U \subset V \Leftrightarrow U \subseteq V \wedge U \neq V$. In [4] it was shown that $(\mathcal{R}_n, \subset) = (\mathcal{R}_m, \subset)$ if and only if $m = n$; also, it was shown that $(\mathcal{R}_n, \subset) = (\mathcal{B}_m, \subset)$ if and only if $m = n = 1$. Using the formulas

$$\mathrm{A}_k := \bigwedge_{1 \leq i \leq k} \Diamond p_i \;\to \bigvee_{1 \leq i < j \leq k} \Diamond(\Diamond p_i \wedge \Diamond p_j), \quad k \geq 2,$$

proposed in [4], we give another example.

For $m \geq 3$, let $\mathcal{P}ol_2^m$ be the set of all convex n-gons on the plane $\mathbb{R}^2$. It is easy to show that the axiom A_k is valid in the frame $(\mathcal{P}ol_2^m, \subset)$ if and only if we have

$$\forall u\; \forall u_1 \ldots \forall u_k \left(\bigwedge_{1 \leq i \leq k} u \subset u_i \to \bigvee_{1 \leq i < j \leq k} \exists v\; (u \subset v \wedge v \subset u_i \wedge v \subset u_j) \right);$$

by straightforward arguments, this property holds for the frame $(\mathcal{P}ol_2^m, \subset)$ if $k > 2m$ and is violated if $k \leq 2m$. Thus, the logics $\mathrm{L}(\mathcal{P}ol_2^m, \subset)$ are different for all m. Analogously, it is possible to distinct logics of polyhedra in a multidimensional space.

Modal axiomatization of the relation $\subset$ (for any set of regions considered in this paper) is unknown. This question is directly related to the unsolved problem put by R. Goldblatt in [11]: to axiomatize the strict causal future relation in the Minkowski space (for details on relativistic modal logics and their relationship with modal logics of regions, see [4]).

The author is grateful to the referee for very detailed and useful comments. Also, the author would like to thank Prof. V. Shehtman for his help during the work on this paper.

REFERENCES

1. Lutz, C. and Wolter, F., Modal Logics of Topological Relations, *Log. Methods Comput. Sci.*, 2006, vol. 2, no. 2, paper 5.
2. Randell, D., Cui, Z., and Cohn, A., A Spatial Logic Based on Regions and Connection, in *Proc. 4th European Workshop on Semantics of Time, Space, and Movement and Spatio-Temporal Reasoning, Château de Bonas, France, 1992*, pp. 391–405.
3. Egenhofer, M. and Franzosa, R., Point-Set Topological Spatial Relations, *Int. J. Geograph. Inform. Syst.*, 1991, vol. 5, no. 2, pp. 161–174.
4. Shapirovsky, I. and Shehtman, V., Modal Logics of Regions and Minkowski Spacetime, *J. Logic Comput.*, 2005, vol. 15, no. 4, pp. 559–574.
5. Chagrov, A. and Zakharyaschev, M., *Modal Logic*, Oxford: Clarendon; New York: Oxford Univ. Press, 1997.
6. Goranko, V. and Passy, S., Using the Universal Modality: Gains and Questions, *J. Logic Comput.*, 1992, vol. 2, no. 1, pp. 5–30.
7. Shapirovsky, I., Downward-Directed Transitive Frames with Universal Relations, *Advances in Modal Logic*, vol. 6, Governatori, G., Hodkinson, I., and Venema, Y., Eds., London: College Publ., 2006, pp. 413–428.
8. Blackburn, P., de Rijke, M., and Venema, Y., *Modal Logic*, Cambridge: Cambridge Univ. Press, 2001.
9. Shapirovsky, I. and Shehtman, V., Chronological Future Modality in Minkowski Spacetime, *Advances in Modal Logic*, vol. 4, Balbiani, P., Suzuki, N.-Y., Wolter, F., and Zakharyaschev, M., Eds., 2003, pp. 437–459.
10. Shapirovsky, I., On PSPACE-Decidability in Transitive Modal Logics, *Advances in Modal Logic*, vol. 5, Schmidt, R., Pratt-Hartmann, I., Reynolds, M., and Wansing, H., Eds., London: College Publ., 2005, pp. 269–287.
11. Goldblatt, R., Diodorean Modality in Minkowski Spacetime, *Studia Logica*, 1980, vol. 39, no. 2–3, pp. 219–236.