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Products of modal logics and tensor products of modal algebras

Dov Gabbay [a,b,c,*], Ilya Shapirovsky [d,*], Valentin Shehtman [d,e,*]

[a] *Department of Computer Science, Bar Ilan University, Israel*
[b] *Department of Informatics, King's College London, United Kingdom*
[c] *Faculty of Science, Technology and Communication, University of Luxembourg, Luxembourg*
[d] *Institute for Information Transmission Problems, Russian Academy of Sciences, Moscow, Russia*
[e] *National Research University Higher School of Economics, Moscow, Russia*

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ABSTRACT

One of natural combinations of Kripke complete modal logics is the product, an operation that has been extensively investigated over the last 15 years. In this paper we consider its analogue for arbitrary modal logics: to this end, we use product-like constructions on general frames and modal algebras. This operation was first introduced by Y. Hasimoto in 2000; however, his paper remained unnoticed until recently. In the present paper we quote some important Hasimoto's results, and reconstruct the product operation in an algebraic setting: the Boolean part of the resulting modal algebra is exactly the tensor product of original algebras (regarded as Boolean rings). Also, we propose a filtration technique for Kripke models based on tensor products and obtain some decidability results.

1. Introduction

Products were introduced in the 1970s as a natural type of combined modal logics. They arise in different areas of pure and applied logic – spatial reasoning, multi-agent systems, quantified modal and intuitionistic logics etc. The theory of products was systematized and essentially developed first in the paper [3] and later in the monograph [4]; during the past 10 years new important results were proved and the research is going on, cf. [7].

Recall that the product of modal logics $\mathrm{L}_1, \mathrm{L}_2$ is defined as the logic of the class of products of their Kripke frames

$$\mathrm{L}_1 \times \mathrm{L}_2 = \mathrm{Log}\big(\{\mathsf{F}_1 \times \mathsf{F}_2 \mid \mathsf{F}_1 \vDash \mathrm{L}_1,\ \mathsf{F}_2 \vDash \mathrm{L}_2\}\big),$$

and the frame $\mathsf{F}_1 \times \mathsf{F}_2$ inherits the horizontal relations from F_1 and the vertical relations from F_2.

On the one hand, this definition is quite natural, and in some cases products can be simply axiomatized and have nice properties.

* Corresponding authors.

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On the other hand, the product operation has some peculiarities.

First, products are always Kripke complete. However, Kripke semantics sometimes may be inadequate. This means that different logics $\mathrm{L}_1, \mathrm{L}_1'$ can have the same frames; in that case $\mathrm{L}_1 \times \mathrm{L}_2 = \mathrm{L}_1' \times \mathrm{L}_2$ for any L_2 – which looks strange.

Second, the product operation is not logically invariant: it may happen that for some frames $\mathrm{Log}(\mathsf{F}) = \mathrm{Log}(\mathsf{F}')$, while $\mathrm{Log}(\mathsf{F} \times \mathsf{G}) \neq \mathrm{Log}(\mathsf{F}' \times \mathsf{G})$. Note that on the contrary, logical invariance holds for direct products of classical models: $Th(M) = Th(M')$ implies $Th(M \times N) = Th(M' \times N)$ (A. Mostowski, 1952; cf. [8, Theorem 19]).

Third, the product of consistent logics L_1 and L_2 may be inconsistent. This happens if L_1 does not have frames (which is possible in the polymodal case, cf. [10], [5, p. 55]).

To amend the situation, one can try to define products of general frames or, equivalently, modal algebras. The following problem was mentioned in [7, p. 877]:

There are several attempts for extending the product construction from Kripke complete logics to arbitrary modal logics, mainly by considering product-like constructions on Kripke models. All the suggested methods so far result in sets of formulas that are not closed under the rule of Substitution.

Nevertheless, a possible answer was already known by that time: it was given by Y. Hasimoto who introduced so called *shifted products* of general frames and modal algebras [6]. In the present paper we revisit this operation. We call it the *tensor product*, since it acts exactly as tensor multiplication on the Boolean parts of modal algebras (regarded as Boolean rings). These shifted (tensor) products are known to be logically invariant and enjoy other nice properties [6].

The paper is organized as follows. Section 2 contains some basic definitions from modal logic. The correlation between modal products of frames and tensor products of Boolean algebras is established in Section 3. Section 4 recalls some basic properties of tensor products stated in [6]. In Section 5 we propose a filtration technique for Kripke models over tensor products and obtain some decidability results.

2. Preliminaries

We assume that the reader is familiar with basic notions in modal logic (see e.g. [1,4]). We recall some of them, mainly for the sake of notation.

An *n-modal algebra* is a Boolean algebra with additional unary operations $\Diamond_1, \dots, \Diamond_n$ (*modalities*) such that $\Diamond_i 0 = 0$ and $\Diamond_i(x \vee y) = \Diamond_i x \vee \Diamond_i y$ for all i (if $n = 1$, we omit the subscript '1'). Fix a countable set of propositional variables $PV = \{p_1, p_2, \dots\}$; ML_n denotes the set of all *n-modal formulas*, i.e., terms over PV in the signature of n-modal algebras.

The notation $A \vDash \varphi$ means that a formula φ is *valid in* an algebra A, i.e., $\varphi = 1$ is true in A under any assignment of propositional variables; φ is *valid in a class* of algebras $\mathfrak{A}$ (in symbols, $\mathfrak{A} \vDash \varphi$) if it is valid in every algebra from $\mathfrak{A}$. The set of all formulas valid in an algebra A is called *the logic of* A and denoted by $\mathrm{Log}(A)$. For a set of formulas Ψ, a *Ψ-algebra* is an algebra A validating all formulas from Ψ (in symbols, $A \vDash \Psi$).

Normal propositional n-modal logics can be defined syntactically or alternatively, as logics of n-modal algebras [1].

A *Kripke n-frame* is a tuple $\mathsf{F} = (W, R_1, \dots, R_n)$, where R_i are binary relations on a nonempty set W. The *modal algebra* of F (denoted by $MA(\mathsf{F})$) is obtained from the Boolean algebra 2^W of all subsets of a set W by expansion with the operations R_i^{-1}, $i = 1, \dots, n$ such that for any set $U \subseteq W$, $R_i^{-1}(U) := \{y \mid \exists x \in U \; yR_i x\}$; cf. [1]. The *logic of* F (in symbols, $\mathrm{Log}(\mathsf{F})$) can be defined as $\mathrm{Log}(MA(\mathsf{F}))$.

A general n-frame is a tuple $\mathsf{F} = (W, R_1, \dots, R_n, A)$, where $(W, R_1, \dots, R_n)$ is a Kripke frame and A is a subalgebra of $MA(W, R_1, \dots, R_n)$. The logic of A is also called *the logic of* F and denoted by $\mathrm{Log}(\mathsf{F})$. *A valuation in* F is a valuation in A, i.e., a map $PV \longrightarrow A$.

A pair $\mathsf{M} = (\mathsf{F}, \theta)$, where θ is a valuation in F, is called *a (Kripke) model over* F. $\|\varphi\|_{\mathsf{M}}$ denotes the value of a modal formula φ in A under θ. The notation $\mathsf{M}, x \vDash \varphi$ means $x \in \|\varphi\|_{\mathsf{M}}$, so in particular

$$\mathsf{M}, x \vDash \Diamond_i \varphi \quad \Leftrightarrow \quad \exists y (x R_i y \ \& \ \mathsf{M}, y \vDash \varphi).$$

A formula φ is called *true in* M if $\|\varphi\|_{\mathsf{M}} = W$; thus φ is valid in F if it is true in all models over F.

Note that a Kripke frame F can be identified with the general frame $(\mathsf{F}, MA(\mathsf{F}))$, and thus for any general frame (F, A), $\mathrm{Log}(\mathsf{F}) \subseteq \mathrm{Log}(\mathsf{F}, A)$.

For a logic L let $\mathrm{Fr}(\mathrm{L})$, $\mathrm{Fr}_{fin}(\mathrm{L})$, $\mathrm{GFr}(\mathrm{L})$ and $\mathrm{Alg}(\mathrm{L})$ be the classes of all Kripke L-frames, finite L-frames, general L-frames and L-algebras respectively.

For a class of n-modal algebras $\mathfrak{C}$, the *logic of* $\mathfrak{C}$ is defined as $\bigcap_{A \in \mathfrak{C}} \mathrm{Log}(A)$ and denoted by $\mathrm{Log}(\mathfrak{C})$; similarly for classes of Kripke frames or general Kripke frames. Logics of Kripke frames are called *Kripke complete*. Note that L is Kripke complete iff $\mathrm{L} = \mathrm{Log}(\mathrm{Fr}(\mathrm{L}))$.

The product (in our terminology, *the modal product*) of Kripke frames $\mathsf{F}_1, \mathsf{F}_2$ is denoted by $\mathsf{F}_1 \times \mathsf{F}_2$. Recall its definition for 1-frames $\mathsf{F}_i = (W_i, R_i)$:

$$\mathsf{F}_1 \times \mathsf{F}_2 = (W_1 \times W_2, R_1^\times, R_2^\times),$$

where

$$\begin{aligned} (w_1, w_2) R_1^\times (v_1, v_2) \quad &\Leftrightarrow \quad w_1 R_1 v_1 \ \& \ w_2 = v_2, \\ (w_1, w_2) R_2^\times (v_1, v_2) \quad &\Leftrightarrow \quad w_1 = v_1 \ \& \ w_2 R_2 v_2. \end{aligned}$$

For classes of Kripke frames $\mathfrak{F}$, $\mathfrak{G}$,

$$\mathfrak{F} \times \mathfrak{G} := \{\mathsf{F} \times \mathsf{G} \mid \mathsf{F} \in \mathfrak{F},\ \mathsf{G} \in \mathfrak{G}\}.$$

For logics $\mathrm{L}_1, \mathrm{L}_2$,

$$\mathrm{L}_1 \times \mathrm{L}_2 := \mathrm{Log}\big(\mathrm{Fr}(\mathrm{L}_1) \times \mathrm{Fr}(\mathrm{L}_2)\big), \qquad \mathrm{L}_1 \times_{fin} \mathrm{L}_2 := \mathrm{Log}\big(\mathrm{Fr}_{fin}(\mathrm{L}_1) \times \mathrm{Fr}_{fin}(\mathrm{L}_2)\big).$$

A logic L has *the finite model property* (*fmp*, for short) if L is the logic of a class of finite frames, or equivalently $\mathrm{L} = \mathrm{Log}(\mathrm{Fr}_{fin}(\mathrm{L}))$. A logic $\mathrm{L}_1 \times \mathrm{L}_2$ has the *product fmp* if $\mathrm{L}_1 \times \mathrm{L}_2 = \mathrm{L}_1 \times_{fin} \mathrm{L}_2$.

3. Tensor products and chequered valuations

It is well known that every Boolean algebra can be regarded as a Boolean ring, where the ring multiplication is the meet and the ring addition is the symmetric difference:

$$xy := x \wedge y, \qquad x + y := (x \wedge \neg y) \vee (y \wedge \neg x).$$

A Boolean ring is a commutative associative algebra over the two-element field $\mathbf{F}_2$ with an idempotent multiplication; so the standard construction of a tensor product of associative algebras is applicable here [9].

Viz., the *tensor product* of algebras $A,\ B$ is a pair $(A \otimes B, \pi)$, where $A \otimes B$ is an algebra, $\pi : (a, b) \mapsto a \otimes b$ is a bilinear map $A \times B \longrightarrow A \otimes B$ with the following universal property: every bilinear map $f : A \times B \longrightarrow C$, where C is an $\mathbf{F}_2$-space, uniquely factors through π, i.e., $f = g \cdot \pi$ for a unique linear $g : A \otimes B \longrightarrow C$.

Note that in the case of vector spaces over $\mathbf{F}_2$ the linearity is expressed by the condition $g(a + b) = g(a) + g(b)$, and the bilinearity of π means $\pi(a + b, c) = \pi(a, c) + \pi(b, c)$, $\pi(a, b + c) = \pi(a, b) + \pi(a, c)$.

The elements of the form $a \otimes b$ linearly generate $A \otimes B$; in our case this means that every element of $A \otimes B$ can be presented as a sum $a_1 \otimes b_1 + \ldots + a_n \otimes b_n$.

The multiplication in $A \otimes B$ is defined in such a way that

$$(a \otimes b)(c \otimes d) = ac \otimes bd.$$

Proposition 3.1. *The tensor product of Boolean rings is a Boolean ring.*

For the proof note that in $A \otimes B$ we have $(a \otimes b)^2 = a^2 \otimes b^2 = a \otimes b$ and $(c_1 + \ldots + c_n)^2 = c_1^2 + \ldots + c_n^2$.

Our aim is to define tensor products of modal algebras. To this end, let us describe the tensor product construction for Boolean rings more explicitly.

Definition 3.2. A set $U \times V$, where $U \subseteq X, V \subseteq Y$, is called a *rectangle* in $X \times Y$. A *chequered subset* of $X \times Y$ is a finite union of rectangles.

Proposition 3.3. *The set of all chequered subsets of $W_1 \times W_2$ is a Boolean subalgebra of $2^{W_1 \times W_2}$. Moreover, if A_i is a subalgebra of 2^{W_i}, $i = 1, 2$, then the set of all finite unions of rectangles $V_1 \times V_2$, where $V_i \in A_i$, is also a Boolean subalgebra of $2^{W_1 \times W_2}$.*

Proof. First note that the complement of a rectangle is chequered:

$$(W_1 \times W_2) - (V_1 \times V_2) = \bigl(W_1 \times (W_2 - V_2)\bigr) \cup \bigl((W_1 - V_1) \times W_2\bigr).$$

Next, note that the intersection of two rectangles is a rectangle:

$$(V_1 \times V_2) \cap (U_1 \times U_2) = (V_1 \cap U_1) \times (V_2 \cap U_2).$$

Hence by distributivity the intersection of chequered subsets is chequered, and thus by De Morgan's law the complement of a chequered subset is chequered.

The above argument goes through also for the case of subalgebras of 2^{W_i}. $\square$

For nonempty sets X, Y let $ch(X, Y)$ be the Boolean algebra of all chequered subsets of $X \times Y$. There is a canonical map $\pi : 2^X \times 2^Y \longrightarrow ch(X, Y)$ such that $\pi(U, V) = U \times V$.

Theorem 3.4. *Let X, Y be nonempty sets. Then $(ch(X, Y), \pi)$ is a tensor product of 2^X and 2^Y. Moreover, let A, B be subalgebras of 2^X, 2^Y respectively, $ch_{AB}(X, Y)$ the Boolean algebra of finite unions of rectangles $U \times V$, where $U \in A$, $V \in B$. Then $(ch_{AB}(X, Y), \pi|(A \times B))$ is a tensor product of Boolean algebras A and B.*

Therefore in the algebra $ch_{AB}(X, Y) = A \otimes B$ every element $U \otimes V$ is just the rectangle $U \times V$.

Proof. One can easily check that π is bilinear. E.g., for the symmetric difference $+$, we have

$$\pi(V_1 + V_2, U) = (V_1 + V_2) \times U = (V_1 \times U) + (V_2 \times U) = \pi(V_1, U) + \pi(V_2, U).$$

To show the universal property of π we need some more definitions and lemmas.

Suppose $\sim_1$ and $\sim_2$ are equivalence relations on X and Y respectively, such that the quotient sets $X/{\sim_1}$ and $Y/{\sim_2}$ are finite. Then the pair $(\sim_1, \sim_2)$ is called a *granulation* on $X \times Y$. It is associated with the following equivalence relation $\sim$ on $X \times Y$:

$$(x, y) \sim (x', y') \ := \ x \sim_1 x' \ \& \ y \sim_2 y';$$

its equivalence classes are called *granules*. Note that granules are rectangles in $X \times Y$. A chequered set is *absorbed by* a granulation if it is a union of granules.

Lemma 3.5. *Every chequered set is absorbed by some granulation. Moreover, if* $S = \bigcup_{i\in I} r_i$*,* I *is finite,* $r_i = U_i \times V_i$*,* $U_i \in A$*,* $V_i \in B$*, then* S *is a disjoint union of a finite set of rectangles* $U \times V$*, where* $U \in A$*,* $V \in B$*.*

Proof. Consider the following equivalence relations on X and Y:

$$x \sim_1 x' := \forall i \in I\bigl(x \in U_i \quad \Leftrightarrow \quad x' \in U_i\bigr);$$
$$y \sim_2 y' := \forall i \in I\bigl(y \in V_i \quad \Leftrightarrow \quad y' \in V_i\bigr).$$

Since I is finite, $X/{\sim_1}$ and $Y/{\sim_2}$ are finite. Note that $X/{\sim_1} \subseteq A$, $Y/{\sim_2} \subseteq B$. Let $\sim$ be the corresponding granulation. Let $(x,y)_\sim$ denote the granule of (x,y). If $(x,y) \in r_i$ then $(x,y)_\sim \subseteq r_i$; thus $r_i = \bigcup_{(x,y)\in r_i} (x,y)_\sim$. $\square$

Now for a bilinear $f : A \times B \longrightarrow C$ let us show that there exists a unique linear $g : ch_{AB}(X,Y) \longrightarrow C$ such that $f = g \cdot \pi$.

1. Uniqueness. By Lemma 3.5, any chequered set is a disjoint union (and so a sum) of rectangles. Thus if two linear maps coincide on rectangles, they must be equal, i.e., $g_1(\pi(U,V)) = g_2(\pi(U,V))$ implies $g_1 = g_2$.

2. Existence. Suppose a granulation $\sim$ absorbs a chequered set S. Put

$$g_\sim(S) := \sum \bigl\{ \bar{f}(r) \bigm| r \in (X \times Y)/{\sim},\ r \subseteq S \bigr\},$$

where $\bar{f}(U \times V) = f(U,V)$. Let us show that $g_\sim(S)$ does not depend on the choice of $\sim$.

Lemma 3.6. *If* $\sim$ *absorbs a rectangle* $U \times V$*, then*

$$f(U,V) = \sum \bigl\{ f\bigl(U',V'\bigr) \bigm| U' \in U/{\sim_1},\ V' \in V/{\sim_2} \bigr\}.$$

Proof. Let $U/{\sim_1} = \{U_1, \dots, U_n\}$, $V/{\sim_2} = \{V_1, \dots, V_m\}$. Then $U = \sum U_i$, $V = \sum V_j$, and by bilinearity,

$$f(U,V) = \sum_{i,j} f(U_i, V_j). \qquad \square$$

A granulation induced by $(\approx_1, \approx_2)$ is *finer* than a granulation induced by $(\sim_1, \sim_2)$, if $\sim_1 \supseteq \approx_1$ and $\sim_2 \supseteq \approx_2$.

Lemma 3.7. *If* $\approx$ *is finer than* $\sim$*, then* $g_\approx(S) = g_\sim(S)$*.*

Proof. $\approx$ absorbs every $\sim$-granule r, so

$$\bar{f}(r) = \sum \bigl\{ \bar{f}\bigl(r'\bigr) \bigm| r' \subseteq r, r' \text{ is a } {\approx}\text{-granule} \bigr\}.$$

Hence by Lemma 3.6,

$$\sum \bigl\{ \bar{f}(r) \bigm| r \subseteq S, r \text{ is a } {\sim}\text{-granule} \bigr\} = \sum \bigl\{ \bar{f}\bigl(r'\bigr) \bigm| r' \subseteq S, r' \text{ is a } {\approx}\text{-granule} \bigr\}. \qquad \square$$

Now given a granulation $\sim$ induced by $(\sim_1, \sim_2)$ and a granulation $\sim'$ induced by $(\sim_1', \sim_2')$, consider the granulation $\approx$ induced by $(\sim_1 \cap \sim_1', \sim_2 \cap \sim_2')$. Then by Lemma 3.7 $g_\sim(S) = g_\approx(S) = g_{\sim'}(S)$.

It follows that $g_\sim(S)$ does not depend on $\sim$, so we can define $g(S)$ as $g_\sim(S)$ for any $\sim$ absorbing S. To check that g is linear, consider chequered sets S_1 an S_2. Let $\sim$ be a granulation absorbing them and let

$$G_i := \{ r \mid r \subseteq S_i, r \text{ is a } {\sim}\text{-granule} \},$$

$i = 1, 2$. Then $S_1 + S_2 = \sum\{r \mid r \in G_1 + G_2\}$. On the other hand,

$$g(S_1) + g(S_2) = \sum_{r \in G_1} \bar{f}(r) + \sum_{r \in G_2} \bar{f}(r) = \sum_{r \in G_1 + G_2} \bar{f}(r) + \sum_{r \in G_1 \cap G_2} \left(\bar{f}(r) + \bar{f}(r)\right)$$
$$= \sum_{r \in G_1 + G_2} \bar{f}(r) = g(S_1 + S_2). \qquad \square$$

Proposition 3.8. *Consider Kripke frames* $\mathsf{F}_1 = (W_1, R_1)$, $\mathsf{F}_2 = (W_2, R_2)$ *and their product* $\mathsf{F}_1 \times \mathsf{F}_2 = (W_1 \times W_2, R_1^\times, R_2^\times)$. *Then*

(1) *for any rectangle* $U \times V$ *we have*

$$R_1^{\times\,-1}(U \times V) = R_1^{-1}(U) \times V, \qquad R_2^{\times\,-1}(U \times V) = U \times R_2^{-1}(V);$$

(2) *if* (F_1, A_1) *and* (F_2, A_2) *are general 1-frames, then* $(\mathsf{F}_1 \times \mathsf{F}_2, A_1 \otimes A_2)$ *is a general 2-frame. In particular,* $(\mathsf{F}_1 \times \mathsf{F}_2, ch(W_1, W_2))$ *is a general 2-frame.*

Proof. (1) readily follows from the definition of $R_1^\times, R_2^\times$. (2) follows from (1) and Theorem 3.4: $A_1 \otimes A_2$ consists of finite unions of rectangles, so it is closed under $R_1^{\times\,-1}$, $R_2^{\times\,-1}$. $\square$

Definition 3.9. The frame $(\mathsf{F}_1 \times \mathsf{F}_2, A_1 \otimes A_2)$ is called the *tensor product of general frames* (F_1, A_1) and (F_2, A_2):

$$(\mathsf{F}_1, A_1) \otimes (\mathsf{F}_2, A_2) := (\mathsf{F}_1 \times \mathsf{F}_2, A_1 \otimes A_2).$$

In particular, the *tensor product of Kripke frames* $\mathsf{F}_1 = (W_1, R_1)$, $\mathsf{F}_2 = (W_2, R_2)$ is

$$\mathsf{F}_1 \otimes \mathsf{F}_2 := \left(\mathsf{F}_1 \times \mathsf{F}_2, ch(W_1, W_2)\right).$$

Tensor products of Kripke frames are also called *chequered frames*.

Theorem 3.10. *If* $(A_1, \Diamond_1)$, $(A_2, \Diamond_2)$ *are normal 1-modal algebras, then there exists a unique 2-modal algebra structure on* $A_1 \otimes A_2$ *with diamond operations* $\Diamond_1^\times, \Diamond_2^\times$ *such that for any* $a \in A_1$, $b \in A_2$

$$\Diamond_1^\times(a \otimes b) = \Diamond_1 a \otimes b, \qquad \Diamond_2^\times(a \otimes b) = a \otimes \Diamond_2 b. \qquad (*)$$

Proof. Due to the Jónsson–Tarski representation theorem [1, Theorem 8.24], each A_i can be identified with an algebra of a general frame; then $\Diamond_i = R_i^{-1}$. The operations $\Diamond_i^\times$ are described by Proposition 3.8(1). Since $a \otimes b = a \times b$ (Theorem 3.4), we obtain $(*)$.

It remains to note that $(*)$ defines unique diamond operations on $A_1 \otimes A_2$, since every element is a sum of rectangles $a \otimes b$. $\square$

For classes of algebras (general frames) $\mathfrak{A}$, $\mathfrak{B}$, put

$$\mathfrak{A} \otimes \mathfrak{B} := \{A \otimes B \mid A \in \mathfrak{A},\ B \in \mathfrak{B}\}.$$

Definition 3.11. The *tensor product* of logics L_1 and L_2 is the logic

$$\mathrm{L}_1 \otimes \mathrm{L}_2 := \mathrm{Log}\left(\mathrm{Alg}(\mathrm{L}_1) \otimes \mathrm{Alg}(\mathrm{L}_2)\right).$$

Since every modal algebra is an algebra of a general frame, we have

$$\mathrm{L}_1 \otimes \mathrm{L}_2 = \mathrm{Log}\big(\mathrm{GFr}(\mathrm{L}_1) \otimes \mathrm{GFr}(\mathrm{L}_2)\big).$$

The next propositions readily follow from this definition.

Proposition 3.12. $\mathrm{L}_1 \otimes \mathrm{L}_2$ *is consistent iff* L_1 *and* L_2 *are consistent.*

Proposition 3.13. *If* L_1 *and* L_2 *are consistent, then* $\mathrm{L}_1 \otimes \mathrm{L}_2$ *is conservative over* L_1 *and* L_2*.*

Proposition 3.14. *If* $\mathrm{L}_1 \otimes \mathrm{L}_2$ *is consistent and Kripke complete, then* L_1 *and* L_2 *are Kripke complete.*

4. Logical invariance

This section contains some important basic properties of tensor products. The results of this section were obtained in [6].

Theorem 4.1. *Suppose* $\mathfrak{A}$, $\mathfrak{A}'$, $\mathfrak{B}$, $\mathfrak{B}'$ *are classes of 1-modal algebras and* $\mathrm{Log}(\mathfrak{A}) = \mathrm{Log}(\mathfrak{A}')$, $\mathrm{Log}(\mathfrak{B}) = \mathrm{Log}(\mathfrak{B}')$*. Then* $\mathrm{Log}(\mathfrak{A} \otimes \mathfrak{B}) = \mathrm{Log}(\mathfrak{A}' \otimes \mathfrak{B}')$*.*

Proof. By a straightforward argument, for any 1-modal algebras A, A', B, B' we have: if $A \in \mathrm{HSP}(A')$, then $A \otimes B \in \mathrm{HSP}(A' \otimes B)$; if $B \in \mathrm{HSP}(B')$, then $A \otimes B \in \mathrm{HSP}(A \otimes B')$, see [6] for more details.

Now we can show that $\mathrm{Log}(\mathfrak{A}) \subseteq \mathrm{Log}(\mathfrak{A}')$ only if $\mathrm{Log}(\mathfrak{A}\otimes\mathfrak{B}) \subseteq \mathrm{Log}(\mathfrak{A}'\otimes\mathfrak{B})$. In fact, by Birkhoff's theorem, $\mathrm{Log}(\mathfrak{A}) \subseteq \mathrm{Log}(\mathfrak{A}')$ implies $\mathfrak{A}' \subseteq \mathrm{HSP}(\mathfrak{A})$, so $\mathfrak{A}' \otimes \mathfrak{B} \subseteq \mathrm{HSP}(\mathfrak{A} \otimes \mathfrak{B})$; hence $\mathrm{Log}(\mathfrak{A} \otimes \mathfrak{B}) \subseteq \mathrm{Log}(\mathfrak{A}' \otimes \mathfrak{B})$.

Thus $\mathrm{Log}(\mathfrak{A}) = \mathrm{Log}(\mathfrak{A}')$ implies $\mathrm{Log}(\mathfrak{A} \otimes \mathfrak{B}) = \mathrm{Log}(\mathfrak{A}' \otimes \mathfrak{B})$, and similarly $\mathrm{Log}(\mathfrak{B}) = \mathrm{Log}(\mathfrak{B}')$ implies $\mathrm{Log}(\mathfrak{A}' \otimes \mathfrak{B}) = \mathrm{Log}(\mathfrak{A}' \otimes \mathfrak{B}')$. $\square$

Corollary 4.2. *Let* $\mathfrak{C}_1$, $\mathfrak{C}_2$ *be classes of 1-modal algebras or general frames,* $\mathrm{L}_1 = \mathrm{Log}(\mathfrak{C}_1)$, $\mathrm{L}_2 = \mathrm{Log}(\mathfrak{C}_2)$*. Then* $\mathrm{L}_1 \otimes \mathrm{L}_2 = \mathrm{Log}(\mathfrak{C}_1 \otimes \mathfrak{C}_2)$*. In particular, if* L_1 *and* L_2 *are Kripke complete then for any classes of Kripke frames* $\mathfrak{F}_1, \mathfrak{F}_2$ *such that* $\mathrm{L}_i = \mathrm{Log}(\mathfrak{F}_i)$*,* $i = 1, 2$*, we have that* $\mathrm{L}_1 \otimes \mathrm{L}_2 = \mathrm{Log}(\mathfrak{F}_1 \otimes \mathfrak{F}_2)$ *is a logic of a class of chequered frames.*

F_L denotes the canonical frame of a consistent logic L, and $(\mathsf{F}_\mathrm{L}, A_\mathrm{L})$ denotes its general canonical frame [1]. It is well known that $\mathrm{L} = \mathrm{Log}(\mathsf{F}_\mathrm{L}, A_\mathrm{L})$. Hence we obtain

Corollary 4.3. *For any consistent* $\mathrm{L}_1,\ \mathrm{L}_2$*,*

$$\mathrm{L}_1 \otimes \mathrm{L}_2 = \mathrm{Log}\big((\mathsf{F}_{\mathrm{L}_1}, A_{\mathrm{L}_1}) \otimes (\mathsf{F}_{\mathrm{L}_2}, A_{\mathrm{L}_2})\big).$$

Recall that a logic L is called *canonical* if $L = \mathrm{Log}(F_L)$.

Corollary 4.4. *If* $\mathrm{L}_1,\ \mathrm{L}_2$ *are canonical, then*

$$\mathrm{L}_1 \otimes \mathrm{L}_2 = \mathrm{Log}(\mathsf{F}_{\mathrm{L}_1} \otimes \mathsf{F}_{\mathrm{L}_2}).$$

Proposition 4.5. $ch(X, Y) = 2^{X\times Y}$ *iff* X *or* Y *is finite.*

Proof. Suppose X is finite. Every $Z \subseteq X \times Y$ can be presented as $\bigcup_{x\in X}((\{x\} \times Y) \cap Z)$. Since every set $(\{x\} \times Y) \cap Z$ is a rectangle, it follows that Z is chequered. $\square$

Corollary 4.6. *If* $\mathrm{L}_1,\ \mathrm{L}_2$ *are Kripke complete, then*

$$\mathrm{L}_1 \times \mathrm{L}_2 \subseteq \mathrm{L}_1 \otimes \mathrm{L}_2 \subseteq \mathrm{L}_1 \times_{\mathit{fin}} \mathrm{L}_2.$$

Proof. By completeness we have $\mathrm{L}_i = \mathrm{Log}(\mathrm{GFr}(\mathrm{L}_i)) = \mathrm{Log}(\mathrm{Fr}(\mathrm{L}_i))$; hence by Theorem 4.1, $\mathrm{L}_1 \otimes \mathrm{L}_2 = \mathrm{Log}(\mathrm{GFr}(\mathrm{L}_1) \otimes \mathrm{GFr}(\mathrm{L}_2)) = \mathrm{Log}(\mathrm{Fr}(\mathrm{L}_1) \otimes \mathrm{Fr}(\mathrm{L}_2))$. The latter logic contains $\mathrm{L}_1 \times \mathrm{L}_2 = \mathrm{Log}(\mathrm{Fr}(\mathrm{L}_1) \times \mathrm{Fr}(\mathrm{L}_2))$, since $\mathrm{Fr}(\mathrm{L}_1) \otimes \mathrm{Fr}(\mathrm{L}_2)$ consists of (some) general frames over frames from $\mathrm{Fr}(\mathrm{L}_1) \times \mathrm{Fr}(\mathrm{L}_2)$.

Next, by Proposition 4.5 we can identify $\mathrm{Fr}_{\mathit{fin}}(\mathrm{L}_1) \times \mathrm{Fr}_{\mathit{fin}}(\mathrm{L}_2)$ with $\mathrm{Fr}_{\mathit{fin}}(\mathrm{L}_1) \otimes \mathrm{Fr}_{\mathit{fin}}(\mathrm{L}_2)$. Thus $\mathrm{L}_1 \otimes \mathrm{L}_2 = \mathrm{Log}(\mathrm{Fr}(\mathrm{L}_1) \otimes \mathrm{Fr}(\mathrm{L}_2)) \subseteq \mathrm{Log}(\mathrm{Fr}_{\mathit{fin}}(\mathrm{L}_1) \otimes \mathrm{Fr}_{\mathit{fin}}(\mathrm{L}_2)) = \mathrm{Log}(\mathrm{Fr}_{\mathit{fin}}(\mathrm{L}_1) \times \mathrm{Fr}_{\mathit{fin}}(\mathrm{L}_2)) = \mathrm{L}_1 \times_{\mathit{fin}} \mathrm{L}_2$. $\square$

Corollary 4.7. *Let* $\mathrm{L}_1,\ \mathrm{L}_2$ *be Kripke complete logics.* $\mathrm{L}_1 \times \mathrm{L}_2$ *has the product fmp iff*

$$\mathrm{L}_1 \times \mathrm{L}_2 = \mathrm{L}_1 \otimes \mathrm{L}_2 = \mathrm{L}_1 \times_{\mathit{fin}} \mathrm{L}_2;$$

it follows that if $\mathrm{L}_1 \times \mathrm{L}_2$ *has the product fmp, then for any* $\mathfrak{F}_i$ *such that* $\mathrm{L}_i = \mathrm{Log}(\mathfrak{F}_i)$*,* $i = 1, 2$*, we have*

$$\mathrm{L}_1 \times \mathrm{L}_2 = \mathrm{Log}(\mathfrak{F}_1 \times \mathfrak{F}_2).$$

Corollary 4.8. *If* L_1 *and* L_2 *have the fmp, then*

$$\mathrm{L}_1 \otimes \mathrm{L}_2 = \mathrm{L}_1 \times_{\mathit{fin}} \mathrm{L}_2,$$

$$\mathrm{L}_1 \times \mathrm{L}_2 \text{ has the product fmp iff } \mathrm{L}_1 \times \mathrm{L}_2 = \mathrm{L}_1 \otimes \mathrm{L}_2.$$

It follows from Theorem 4.1 that usually modal and tensor products lead to different logics. In fact, modal products of logics with the fmp in many cases do not have the product fmp [4]. Rare exceptions are $\mathbf{K} \times \mathbf{K}$ and $\mathbf{S5} \times \mathbf{S5}$ having the product fmp [3,4]. Also, $\mathbf{K} \times \mathbf{K} = [\mathbf{K}, \mathbf{K}]$, where $[\mathrm{L}_1, \mathrm{L}_2]$ denotes the *commutator* of L_1 and L_2 [4, p. 378].

Corollary 4.9. $[\mathrm{L}_1, \mathrm{L}_2] \subseteq \mathrm{L}_1 \otimes \mathrm{L}_2$*.*

Proof. By Proposition 3.13, $\mathrm{L}_1 \otimes \mathrm{L}_2$ contains the fusion of L_1 and L_2. Also, $\mathrm{L}_1 \otimes \mathrm{L}_2$ contains $\mathbf{K} \otimes \mathbf{K} = [\mathbf{K}, \mathbf{K}]$, so it contains the commutativity and confluence axioms of the commutator. (Note that for the case of Kripke complete logics this follows from the Corollary 4.6 by well-know inclusion $[\mathrm{L}_1, \mathrm{L}_2] \subseteq \mathrm{L}_1 \times \mathrm{L}_2$.) $\square$

5. Filtrations of chequered models

Recall the standard definition of *filtrations of Kripke models* (see e.g. [5, Part I, Section 4]).

Definition 5.1. Let $\mathsf{M} = (W, R, \theta)$ be a Kripke model, Φ a set of formulas.

Consider the equivalence relation $\sim_\Phi$ on W:

$$\sim_\Phi := \bigl\{ (x, x') \bigm| \forall \varphi \in \Phi \ \bigl(\mathsf{M}, x \vDash \varphi \ \Leftrightarrow \ \mathsf{M}, x' \vDash \varphi\bigr) \bigr\}.$$

Let $[x]$ be the equivalence class of x w.r.t. $\sim_\Phi$. Consider relations $R_{\min}$ and $R_{\max}$ on the quotient set $W/{\sim_\Phi}$:

$$R_{\min} := \bigl\{ \bigl([x], [y]\bigr) \bigm| \exists x' \in [x]\, \exists y' \in [y]\ x' R y' \bigr\},$$
$$R_{\max} := \bigl\{ \bigl([x], [y]\bigr) \bigm| \forall \varphi (\Diamond\varphi \in \Phi \ \&\ \mathsf{M}, y \vDash \varphi \ \Rightarrow \ \mathsf{M}, x \vDash \Diamond\varphi) \bigr\}.$$

They are called *the minimal* and *the maximal filtrating relations*. A model

$$\bar{\mathsf{M}} = (W/{\sim_\Phi}, \bar{R}, \bar{\theta})$$

is called a *filtration of* M *through* Φ if for any $p \in \Phi$

$$\bar{\theta}(p) = \{[x] \mid \mathsf{M}, x \vDash p\},$$

and

$$R_{\min} \subseteq \bar{R} \subseteq R_{\max}.$$

For models M_1, M_2, the notation $\mathsf{M}_1, x \sim_\Phi \mathsf{M}_2, y$ means

$$\forall \varphi \in \Phi\, (\mathsf{M}_1, x \vDash \varphi \;\Leftrightarrow\; \mathsf{M}_2, y \vDash \varphi).$$

Lemma 5.2 *(Filtration Lemma). (See [5].) Consider a set of formulas Φ closed under subformulas. If $\bar{\mathsf{M}}$ is a filtration of M through Φ, then for any x in M*

$$\mathsf{M}, x \sim_\Phi \bar{\mathsf{M}}, [x].$$

Definition 5.3. (See [2].) A logic L *admits filtration* if it is Kripke complete and for any L-frame (W, R), for any model $\mathsf{M} = (W, R, \theta)$, and for any finite set of formulas Φ closed under subformulas, there exists a filtration $(\bar{W}, \bar{R}, \bar{\theta})$ of M through Φ such that $(\bar{W}, \bar{R})$ is an L-frame.

It is well known that many logics admit filtration; in particular, this is true for every logic axiomatized by some of the axioms

$$\Box p \to \Box\Box p, \qquad \Box p \to p, \qquad \Diamond\Box p \to p, \qquad \Box p \to \Diamond p, \qquad \Diamond p \to \Diamond\Diamond p$$

[2,5]. So all the logics $\mathbf{K}, \mathbf{K4}, \mathbf{T}, \mathbf{S4}, \mathbf{S5}$ admit filtration.

It follows that if a logic L admits filtration, then it has the fmp. Moreover, to check the L-satisfiability of a formula φ, it is sufficient to consider L-frames of cardinality at most $2^{\#\varphi}$, where $\#\varphi$ denotes the cardinality of the set $sub(\varphi)$ of all subformulas of φ. Our aim is to formulate an analogous result for chequered frames.

Definition 5.4. Consider Kripke frames $\mathsf{F}_1, \mathsf{F}_2$, $\mathsf{F}_i = (W_i, R_i)$, a model $\mathsf{M} = (\mathsf{F}_1 \times \mathsf{F}_2, \theta)$, and a set of formulas Φ. Consider relations $\sim_i$ on W_i:

$$\begin{aligned} \sim_1 &:= \{(x, x') \mid (x, y) \sim_\Phi (x', y) \text{ for all } y \in W_2\}; \\ \sim_2 &:= \{(y, y') \mid (x, y) \sim_\Phi (x, y') \text{ for all } x \in W_1\}. \end{aligned}$$

The pair $(\sim_1, \sim_2)$ is called the *Φ-granulation of* M.

Obviously, $\sim_i$ is an equivalence relation on W_i.

Proposition 5.5. *Consider a model*

$$\mathsf{M} = \bigl((W_1, R_1) \otimes (W_2, R_2), \theta\bigr),$$

a finite set of formulas Φ, and the Φ-granulation $(\sim_1, \sim_2)$. Then the quotient sets $W_1/{\sim_1}$ and $W_2/{\sim_2}$ are finite (i.e., a Φ-granulation is really a granulation).

Proof. The value $\|\varphi\|_{\mathsf{M}}$ of a formula φ in M is a union of a finite set X_φ of rectangles, i.e., $\|\varphi\|_{\mathsf{M}} = \bigcup X_\varphi$.

Let $(\equiv_1, \equiv_2)$ be a granulation absorbing the chequered set X (Lemma 3.5). Let us show that $\equiv_i \subseteq \sim_i$ for $i = 1, 2$. Since $x \equiv_1 x'$, for any y, (x, y) and (x, y') are in the same granule, and thus for any φ, they are in the same rectangles from X_φ. So for any φ,

$$(x, y) \in \|\varphi\|_M \quad \text{iff} \quad (x', y) \in \|\varphi\|_M,$$

which means $(x, y) \sim_\Phi (x', y)$.

Since $W_1/{\equiv_1}$ and $W_2/{\equiv_2}$ are finite, the quotient sets $W_1/{\sim_1}$ and $W_2/{\sim_2}$ are also finite. □

Proposition 5.6. *Suppose* L *admits filtration,* F, G *are Kripke frames,* $\mathsf{F} \vDash \mathrm{L}$*,* φ *is true at some point in a chequered model* $\mathsf{M} = (\mathsf{F} \otimes \mathsf{G}, \theta)$*,* $(\sim_1, \sim_2)$ *is the* $sub(\varphi)$*-granulation of* M*. Then there exists a Kripke frame* $\bar{\mathsf{F}}$ *such that* $\bar{\mathsf{F}} \vDash \mathrm{L}$*,* φ *is satisfiable in* $\bar{\mathsf{F}} \times \mathsf{G}$*, and the cardinality of* $\bar{\mathsf{F}}$ *is not greater than*

$$2^{\#\varphi \cdot |W_2/\sim_2|}.$$

Proof. Let $\mathsf{F} = (H, R)$, $\mathsf{G} = (V, S)$. By Proposition 5.5, $H/{\sim_1}$ and $V/{\sim_2}$ are finite. Suppose $H/{\sim_1} = \{H_1, \dots, H_h\}$, $V/{\sim_2} = \{V_1, \dots, V_v\}$.

Let φ be a formula in variables $p_1, \dots, p_m$, $\Phi = sub(\varphi)$, and let $\Diamond_2\psi_1, \dots, \Diamond_2\psi_k$ be all subformulas of φ of the form $\Diamond_2\psi$.

Fix fresh variables $q_1, \dots, q_k$, $q_1^i, \dots, q_k^i$, $p_1^i, \dots, p_m^i$, $1 \le i \le v$. Consider the translation $g : \Phi \longrightarrow ML_1$, preserving propositional variables, the Boolean connectives, $\Diamond_1$, and such that $g(\Diamond_2\psi_j) := q_j$, $1 \le j \le k$. For $1 \le i \le v$ and for any formula $\psi(p_1, \dots, p_m) \in ML_1$, put

$$f_i(\psi) := \left[q_1^i, \dots, q_k^i, p_1^i, \dots, p_m^i / q_1, \dots, q_k, p_1, \dots, p_m\right]\psi.$$

Thus all subformulas of φ beginning with $\Diamond_2$ are replaced with fresh variables, and then all variables are renamed differently for all horizontal levels ("floors") i.

Now, fix points $y_i \in V_i$, and consider a model $\mathsf{N} = (\mathsf{F}, \eta)$, where

$$\eta\left(p_j^i\right) := \left\{x \mid \mathsf{M}, (x, y_i) \vDash p_j\right\}, \tag{1}$$

$$\eta\left(q_j^i\right) := \left\{x \mid \mathsf{M}, (x, y_i) \vDash \Diamond_2\psi_j\right\}. \tag{2}$$

Claim 1. For all $x \in H$, $1 \le i \le v$, $\psi \in \Phi$,

$$\mathsf{M}, (x, y_i) \vDash \psi \quad \Leftrightarrow \quad \mathsf{N}, x \vDash f_i(g(\psi)). \tag{3}$$

By induction on the length of ψ. If ψ is a variable, then $\psi = p_j$ for some $1 \le j \le m$ and $f_i(g(\psi)) = p_j^i$; in this case (3) follows from (1). If ψ is of the form $\Diamond_2\xi$ then $\psi = \Diamond_2\psi_j$ for some $1 \le j \le k$ and $f_i(g(\psi)) = q_j^i$, so (3) follows from (2). The cases of Boolean connectives and $\Diamond_1$ follows readily from the IH.

Put

$$\Psi := \bigcup_{1 \le i \le v} \left\{f_i\left(g(\psi)\right) \mid \psi \in \Phi\right\}.$$

Note that if α is a subformula of some formula $f_i(g(\psi))$ from Ψ, then $\alpha = f_i(g(\psi'))$, where ψ' is a subformula of ψ, thus $\psi' \in \Phi$. It follows that Ψ is closed under subformulas.

Claim 2. For all $x, x' \in H$,

$$x \sim_1 x' \quad \Leftrightarrow \quad \mathsf{N}, x \sim_\Psi \mathsf{N}, x'.$$

Suppose $x \sim_1 x'$, i.e., for all y in V $\mathsf{M}, (x, y) \sim_\Phi \mathsf{M}, (x', y)$, and let $\alpha \in \Psi$. Then $\alpha = f_i(g(\psi))$ for some $\psi \in \Phi$, $1 \leq i \leq v$. By Claim 1,

$$\mathsf{N}, x \vDash \alpha \quad \Leftrightarrow \quad \mathsf{M}, (x, y_i) \vDash \psi, \qquad \mathsf{M}, \big(x', y_i\big) \vDash \psi \quad \Leftrightarrow \quad \mathsf{N}, x' \vDash \alpha,$$

so $\mathsf{N}, x \vDash \alpha \;\Leftrightarrow\; \mathsf{N}, x' \vDash \alpha$. It follows that $\mathsf{N}, x \sim_\Psi \mathsf{N}, x'$.

The other way round, if $x \nsim_1 x'$, then $\mathsf{M}, (x, y) \nsim_\Phi \mathsf{M}, (x', y)$ for some $y \in V$. Let $y \in V_i$, i.e., $y \sim_2 y_i$; then in M we have $(x, y) \sim_\Phi (x, y_i)$, $(x', y) \sim_\Phi (x', y_i)$, hence, $(x, y_i) \nsim_\Phi (x', y_i)$. Therefore, for some $\psi \in \Phi$ the equivalence $\mathsf{M}, (x, y_i) \vDash \psi \;\Leftrightarrow\; \mathsf{M}, (x', y_i) \vDash \psi$ does not hold. Then by Claim 1 $\mathsf{N}, x \vDash f_i(g(\psi)) \;\not\Leftrightarrow\; \mathsf{N}, x' \vDash f_i(g(\psi))$, which means that $\mathsf{N}, x \nsim_\Psi \mathsf{N}, x'$.

Since L admits filtration, there exists a filtration $\bar{\mathsf{N}} = (\bar{H}, \bar{R}, \bar{\eta})$ of N through Ψ such that $(\bar{H}, \bar{R}) \vDash \mathrm{L}$. By Claim 2, $\bar{H} = H/{\sim_1} = \{H_1, \ldots, H_h\}$.

Put $\bar{\mathsf{F}} := (\bar{H}, \bar{R})$. Let $[x]$ denote the $\sim_1$-class of x. Consider a valuation $\bar{\theta}$ on the Kripke frame $\bar{\mathsf{F}} \times \mathsf{G}$ such that $([x], y) \in \bar{\theta}(p) \Leftrightarrow (x, y) \in \theta(p)$ for all $p \in \Phi$, $y \in V$ (recall that if $x \sim_1 x'$ then $(x, y) \in \theta(p) \Leftrightarrow (x', y) \in \theta(p)$, so $\bar{\theta}$ is well-defined).

Claim 3. For any $(x, y) \in W$, $\psi \in \Phi$

$$\mathsf{M}, (x, y) \vDash \psi \quad \Leftrightarrow \quad (\bar{\mathsf{F}} \times \mathsf{G}, \bar{\theta}), \big([x], y\big) \vDash \psi.$$

By induction on the length of ψ. Consider the only non-trivial case $\psi = \Diamond_1 \xi$.

If $\mathsf{M}, (x, y) \vDash \Diamond_1 \xi$ then $\mathsf{M}, (x', y) \vDash \xi$ and xRx' for some x'. By Definition 5.1 we have $[x]\bar{R}[x']$. By the induction hypothesis,

$$(\bar{\mathsf{F}} \times \mathsf{G}, \bar{\theta}), \big(\big[x'\big], y\big) \vDash \xi,$$

that is

$$(\bar{\mathsf{F}} \times \mathsf{G}, \bar{\theta}), \big([x], y\big) \vDash \Diamond_1 \xi.$$

The proof in the opposite direction is more interesting. Suppose $(\bar{\mathsf{F}} \times \mathsf{G}, \bar{\theta}), ([x], y) \vDash \Diamond_1 \xi$. Then $[x]\bar{R}[x']$ and $(\bar{\mathsf{F}} \times \mathsf{G}, \bar{\theta}), ([x'], y) \vDash \xi$ for some x'. By the induction hypothesis, $\mathsf{M}, (x', y) \vDash \xi$. Then $y \sim_2 y_i$ for some i. Therefore, $\mathsf{M}, (x', y_i) \vDash \xi$. By Claim 1, $\mathsf{N}, x' \vDash f_i(g(\xi))$. By Filtration Lemma, $\bar{\mathsf{N}}, [x'] \vDash f_i(g(\xi))$. Since $[x]\bar{R}[x']$, we obtain $\bar{\mathsf{N}}, [x] \vDash \Diamond_1 f_i(g(\xi))$. Now observe that $\Diamond_1 f_i(g(\xi)) = f_i(g(\Diamond_1 \xi))$, so using Filtration Lemma and Claim 1 again we obtain that $\mathsf{M}, (x, y_i) \vDash \Diamond_1 \xi$. Since $y \sim_2 y_i$, we finally obtain $\mathsf{M}, (x, y) \vDash \Diamond_1 \xi$, which proves the claim.

To complete the proof of the proposition, it remains to estimate the size h of $\bar{\mathsf{F}}$. Clearly, $h \leq 2^{|\Psi|}$. On the other hand, $|\Psi| = \#\varphi \cdot v$ and v is the number of $\sim_2$-classes. □

Unlike the usual filtration technique, the above proposition does not estimate the size of a countermodel, because it gives an upper bound depending on $W_2/{\sim_2}$. However, in some cases it implies decidability results for tensor and modal products.

Tensor and modal products of Kripke complete logics coincide when one of the logics is tabular:

Theorem 5.7. *If* L_1 *is Kripke complete and* L_2 *is tabular, then*

$$\mathrm{L}_1 \times \mathrm{L}_2 = \mathrm{L}_1 \otimes \mathrm{L}_2.$$

Proof. $\mathrm{L}_1 \times \mathrm{L}_2$ is complete with respect to the class

$$\mathfrak{C} = \{\mathsf{F} \times \mathsf{G} \mid \mathsf{F} \vDash \mathrm{L}_1,\ \mathsf{G} \vDash \mathrm{L}_2,\ \mathsf{G} \text{ is rooted}\},$$

see e.g. [4]. If $\mathsf{G} \vDash \mathrm{L}_2$ and G is rooted, then by the tabularity of L_2, G is finite [1]. By Proposition 4.5, all frames in the class $\mathfrak{C}$ are chequered, so $\mathrm{L}_1 \times \mathrm{L}_2 = \mathrm{L}_1 \otimes \mathrm{L}_2$. □

Hence we readily obtain the following properties of modal products with tabular logics.

Corollary 5.8. *For a class of frames* $\mathfrak{F}$*, and a finite frame* G*,*

$$\mathrm{Log}(\mathfrak{F}) \times \mathrm{Log}(\mathsf{G}) = \mathrm{Log}\big(\mathfrak{F} \times \{\mathsf{G}\}\big).$$

Corollary 5.9. *If* L_1 *has the fmp and* L_2 *is tabular, then* $\mathrm{L}_1 \times \mathrm{L}_2$ *has the product fmp.*

Corollary 5.10. *The modal product of tabular logics is tabular: if* F *and* G *are finite, then*

$$\mathrm{Log}(\mathsf{F}) \times \mathrm{Log}(\mathsf{G}) = \mathrm{Log}(\mathsf{F} \times \mathsf{G}).$$

Theorem 5.11. *Suppose* L_2 *is tabular. Then:*

1. *if* L_1 *admits filtration, then* $\mathrm{L}_1 \times \mathrm{L}_2$ *has the exponential product fmp;*
2. *if* L_1 *is Kripke complete then* $\mathrm{L}_1 \times \mathrm{L}_2$ *is m-reducible to* L_1*; so if* L_1 *is Kripke complete and decidable, then* $\mathrm{L}_1 \times \mathrm{L}_2$ *is decidable.*

Proof. 1. Suppose $\mathrm{L}_2 = \mathrm{Log}(\mathsf{G})$ for a finite G of size n. By Proposition 5.6, φ is $\mathrm{L}_1 \times \mathrm{L}_2$-satisfiable iff φ is satisfiable in a frame $\mathsf{F} \times \mathsf{G}$, where $\mathsf{F} \vDash \mathrm{L}_1$ and the size of F is not greater than $2^{\#\varphi \cdot n}$.

2[1]. Let $\mathsf{G} = (\{1, \dots, n\}, S)$.

Take auxiliary propositional variables p_j^i, $1 \le i \le n$, $j \ge 1$. For all i, $1 \le i \le n$, define the translation $g_i : ML_2 \longrightarrow ML_1$ preserving the Boolean connectives, $\Diamond_1$, and such that

$$g_i(p_j) := p_j^i, \qquad g_i(\Diamond_2 \psi) := \bigvee_{iSk} g_k(\psi).$$

Now the second statement of the theorem follows from two lemmas.

Lemma 5.12. *Suppose* $\mathsf{M} = (\mathsf{F} \times \mathsf{G}, \theta)$ *and* $\mathsf{N} = (\mathsf{F}, \eta)$ *are Kripke models such that for all* x *in* F*,* $1 \le i \le n$*,* $j \ge 1$*,*

$$\mathsf{M}, (x, i) \vDash p_j \quad \Leftrightarrow \quad \mathsf{N}, x \vDash p_j^i. \tag{4}$$

Then for any $\varphi \in ML_2$*,* x *in* F*,* $1 \le i \le n$*,*

$$\mathsf{M}, (x, i) \vDash \varphi \quad \Leftrightarrow \quad \mathsf{N}, x \vDash g_i(\varphi). \tag{5}$$

[1] The idea of the proof is similar to an argument by D.P. Skvortsov reducing first-order modal logics with finite constant domains to propositional modal logics.

Proof. By induction on the length of ψ. The induction base coincides with (4). The cases of Boolean connectives and $\Diamond_1$ are trivial.

Consider the case $\varphi = \Diamond_2\xi$. $\mathsf{M}, (x, i) \vDash \Diamond_2\xi$ means that for some k with iSk we have $\mathsf{M}, (x, k) \vDash \xi$; by the IH, the latter is equivalent to $\mathsf{N}, x \vDash g_k(\xi)$; thus

$$\mathsf{M}, (x, i) \vDash \varphi \quad \text{iff} \quad \mathsf{N}, x \vDash \bigvee_{iSk} g_k(\xi) \bigl(= g_i(\Diamond_2\xi)\bigr). \qquad \square$$

Lemma 5.13. *For any frame* F *and a formula* $\varphi \in ML_2$,

$$\varphi \text{ is satisfiable in } \mathsf{F} \times \mathsf{G} \quad \text{iff} \quad g_1(\varphi) \vee \ldots \vee g_n(\varphi) \text{ is satisfiable in } \mathsf{F}.$$

Thus satisfiability in $\mathsf{F} \times \mathsf{G}$ *is m-reducible to satisfiability in* F.

Proof. Suppose φ is satisfiable in $\mathsf{F} \times \mathsf{G}$. Then $\mathsf{M}, (x, i) \vDash \varphi$ for some i, x in F and a model M based on $\mathsf{F} \times \mathsf{G}$. Consider a model N based on F and satisfying (4). Then by (5) we have $\mathsf{N}, x \vDash g_i(\varphi)$. Therefore $g_1(\varphi) \vee \ldots \vee g_n(\varphi)$ is satisfiable in F.

The other way round, if $g_1(\varphi) \vee \ldots \vee g_n(\varphi)$ is satisfiable in F, then $\mathsf{N}, x \vDash g_i(\varphi)$ for some i, x in F and a model N based on F. Consider M based on $\mathsf{F} \times \mathsf{G}$ and satisfying (4); then by (5) we obtain $\mathsf{M}, (x, i) \vDash \varphi$, so φ is satisfiable in $\mathsf{F} \times \mathsf{G}$. $\square$

6. Conclusion

In this paper we have studied properties of a product-like operation $\otimes$ on arbitrary (not only Kripke complete) modal logics introduced in [6]. We have showed that this operation is correlated with tensor products of modal algebras, and proved some completeness and decidability results.

The above considerations can literally be transferred to the polymodal case; in particular, the logical invariance also holds for tensor products of polymodal logics. One of the corollaries is the associativity of tensor products: indeed, for any frames we have

$$\bigl((\mathsf{F}_1, A_1) \otimes (\mathsf{F}_2, A_2)\bigr) \otimes (\mathsf{F}_3, A_3) \cong (\mathsf{F}_1, A_1) \otimes \bigl((\mathsf{F}_2, A_2) \otimes (\mathsf{F}_3, A_3)\bigr),$$

so for any logics we have

$$(\mathrm{L}_1 \otimes \mathrm{L}_2) \otimes \mathrm{L}_3 = \mathrm{L}_1 \otimes (\mathrm{L}_2 \otimes \mathrm{L}_3).$$

Note that the analogous question for modal products of logics is open, see e.g. [7, p. 877].

There are many open questions about tensor products of modal logics. Let us quote some of them.

1. Does Kripke completeness transfer from L_1 and L_2 to $\mathrm{L}_1 \otimes \mathrm{L}_2$?
2. Do there exist $\mathrm{L}_1, \mathrm{L}_2$ such that $\mathrm{L}_1 \times \mathrm{L}_2 = \mathrm{L}_1 \otimes \mathrm{L}_2$, but $\mathrm{L}_1 \times \mathrm{L}_2$ lacks the product fmp and $\mathrm{L}_1, \mathrm{L}_2$ are non-tabular?
3. Do there exist $\mathrm{L}_1, \mathrm{L}_2$ such that $\mathrm{L}_1 \times \mathrm{L}_2$ is undecidable, but $\mathrm{L}_1 \otimes \mathrm{L}_2$ is decidable?
4. Do there exist $\mathrm{L}_1, \mathrm{L}_2$ such that $\mathrm{L}_1 \times \mathrm{L}_2$ is not finitely axiomatizable, but $\mathrm{L}_1 \otimes \mathrm{L}_2$ is finitely axiomatizable?
5. When the inclusions described in Corollaries 4.6, 4.9 are strict? For some cases, the answer follows from Corollaries 4.6, 4.7; in general, this question is open.

Another natural operation on modal algebras is the *normal product* [6]. For monomodal algebras $(A_1, \Diamond_1)$, $(A_2, \Diamond_2)$, their normal product is defined as the monomodal algebra $(A_1 \otimes A_2, \Diamond_1 \cdot \Diamond_2)$. The corresponding binary operation on general frames sends a pair $((\mathsf{F}_1, A_1), (\mathsf{F}_2, A_2))$ to $(\mathsf{G}, A_1 \otimes A_2)$, where G is the direct product of F_1 and F_2 (in the standard model-theoretic sense). Many nice properties of normal products (including logical invariance) were proved in [6]. Also note that in [6] tensor ('shifted') products were defined in an equivalent way via normal products and fusions.

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