Locally finite polymodal logics and Segerberg – Maksimova criterion

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Part I. Preliminaries.

Language

Fix a set $\mathcal{A}$ for the *alphabet of modal operators*.
$\mathcal{A}$-formulas: a countable set $\textsc{Var}$ (propositional variables), Boolean connectives, unary connectives $\Diamond \in \mathcal{A}$.

Normal modal logics: Definition 1

A set of modal $\mathcal{A}$-formulas L is a *normal modal logic*, if for all $\Diamond \in \mathcal{A}$, L contains

classical tautologies

$$\Diamond\bot \leftrightarrow \bot, \quad \Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$$

and is closed under MP, Sub, and *Mon*:
if $(\varphi \to \psi) \in L$, then $(\Diamond\varphi \to \Diamond\psi) \in L$.

Normal modal logics: Definition 2

A *modal algebra* is a Boolean algebra B endowed with unary operations that distributes over finite disjunctions.

A set of modal formulas L is a *normal modal logic*, if L is the logic of a modal algebra B: $L = \{\varphi \mid \varphi = 1 \text{ holds in B}\}$

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Kripke semantics

A *(Kripke) frame* F: $(W, (R_\Diamond)_{\Diamond\in\mathcal{A}})$, where $R_\Diamond$ are binary relations on a set W.
A *model* M on F is a pair (F, θ) where $\theta : \textsc{Var} \to \mathcal{P}(W)$.

$$M, x \vDash p \text{ iff } x \in \theta(p), \qquad M, x \vDash \Diamond\varphi \text{ iff } M, y \vDash \varphi \text{ for some } y \text{ with } xR_\Diamond y.$$

$\mathrm{Log}(\mathrm{F}) = \{\varphi \mid \mathrm{F} \vDash \varphi\}$, where $\mathrm{F} \vDash \varphi$ means that $M, x \vDash \varphi$ for all M on F and all x in M.

The *algebra $\mathrm{Alg}(\mathrm{F})$ of an $\mathcal{A}$-frame* $(W, (R_\Diamond)_{\Diamond\in\mathcal{A}})$ is the powerset algebra of W with the unary operations $f_\Diamond$: for $Y \subseteq W$, $f_\Diamond(Y)$ is $R_\Diamond^{-1}[Y] = \{x \mid \exists y \in Y \, xR_\Diamond y\}$.

$$\mathrm{F} \vDash \varphi \quad \text{iff} \quad \varphi = 1 \text{ holds in } \mathrm{Alg}(\mathrm{F})$$

A logic L is *Kripke complete*, if L is the logic of a class $\mathcal{C}$ of Kripke frames: $L = \bigcap\{\mathrm{Log}(\mathsf{F}) \mid \mathsf{F} \in \mathcal{C}\}$.

A logic L has the *finite model property*, if L is the logic of a class $\mathcal{C}$ of finite frames (algebras, models).

Fact

If L has the fmp and is finitely axiomatizable, then it is decidable.

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An algebra A is *locally finite*, if every finitely generated subalgebra of A is finite.

A logic L is *locally finite* (aka *locally tabular*), if for all $k < \omega$ there are only finitely many formulas in k variables (up to $\leftrightarrow_L$).

TFAE:

L is locally finite.	Every finitely generated free (aka Lindenbaum-Tarski) algebra of L is finite.	The variety of L-algebras is *locally finite*, i.e., all finitely generated L-algebra are finite.

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normal modal logics $\supsetneq$
Kripke complete logics $\supsetneq$
logics with the finite model property $\supsetneq$
logics whose all extensions have the fmp $\supsetneq$
locally finite logics

Local finiteness for modal logics and their relatives (sound but incomplete list):

Segerberg, K., "An Essay in Classical Modal Logic," 1971.

Kuznetsov, A. *Some properties of the structure of varieties of pseudo-Boolean algebras*, 1971.

Maksimova, L. *Modal logics of finite slices*, 1975.

Komori, Y. *The finite model property of the intermediate propositional logics on finite slices*, 1975.

Byrd, M. *On the addition of weakened L-reduction axioms to the Brouwer system*, 1978.

Makinson, D. *Non-equivalent formulae in one variable in a strong omnitemporal modal logic*, 1981.

Mardaev, S. *The number of prelocally tabular superintuitionistic propositional logics*, 1984.

Citkin, A. *Finite axiomatizability of locally tabular superintuitionistic logics*, 1986.

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Bezhanishvili, G. and Grigolia, R. *Locally tabular extensions of MIPC*, 1998.

Bezhanishvili, G., *Locally finite varieties*, 2001.

Bezhanishvili, N. *Varieties of two-dimensional cylindric algebras. Part I: Diagonal-free case*, 2002.

Shehtman, V. *Canonical filtrations and local tabularity*, 2014.

Sh. and Shehtman, V. *Local tabularity without transitivity*, 2016

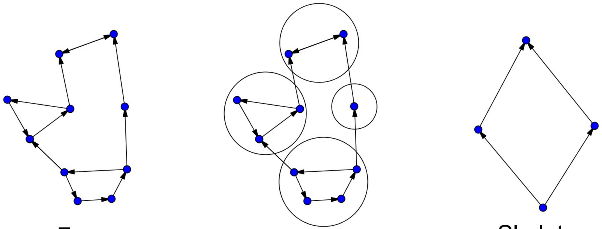
Sh. *Glivenko's theorem, finite height, and local tabularity*, 2021.

Dzik W., Kost S., Wojtylak P. *Finitary unification in locally tabular modal logics characterized*, 2022

Sh. *Sufficient conditions for local tabularity of a polymodal logic*, to appear.

...

A poset is of *finite height* $\leq h$ if its every chain contains at most h elements.



A *cluster* in an $\mathcal{A}$-frame $\mathsf{F} = (W, (R_\Diamond)_{\Diamond \in \mathcal{A}})$ is an equivalence class with respect to the relation

$$\sim_\mathsf{F} = \{(a, b) \mid aR_\mathsf{F} b \text{ and } bR_\mathsf{F} a\},$$

where R_F is the transitive reflexive closure of $\bigcup_\mathcal{A} R_\Diamond$.

For clusters C, D, put

$$C \leq_\mathsf{F} D \text{ iff } xR_\mathsf{F} y \text{ for some (all) } x \in C, y \in D.$$

The poset $(W/{\sim_\mathsf{F}}, \leq_\mathsf{F})$ is called the *skeleton of* F.

The *height of a frame* F is the height of its skeleton.

Unimodal transitive case

$\Diamond\Diamond p \to \Diamond p$ expresses the transitivity of a binary relation.

Formulas of *finite height*:

$$B_0 = \bot, \quad B_{i+1} = p_{i+1} \to \Box(\Diamond p_{i+1} \vee B_i)$$

($\Box$ abbreviates $\neg\Diamond\neg$)

[Segerberg 1971; Maksimova 1975]
Let $L \vdash \Diamond\Diamond p \to \Diamond p$. Then

L is LF iff $L \vdash B_n$ for some n.

This is a nice theorem in many respects. Besides the fact that it gives a natural semantic criterion of local finiteness (in terms of relations), it also provides an **axiomatic characterization**: local finiteness is expressed by a formula from an explicitly described set $\{B_n \mid n < \omega\}$.

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Unimodal nontransitive case

Sometimes an analog of finite height criterion holds in non-transitive case:

[Shehtman & Sh, 2016] Let L be a unimodal logic containing $\Diamond \ldots \Diamond p \to \Diamond p \vee p$. Then

L is LF iff L has a formula of finite height.

(Formulas of finite height will be slight modifications of B_i's.)

But in general, in non-transitive case, finite height is not sufficient:

[Byrd 1978; Makinson, 1981] There is a reflexive symmetric structure $\mathrm{F} = (W, R)$ with $R \circ R = W \times W$ (so its height is 1) s.t. its logic is not LF. Moreover, its one-variable fragment is infinite.

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Polymodal case: (non)examples

If the 1-variable fragment of an $\mathcal{A}$-logic L is finite, then:
(1) [Folklore(?)] For some finite m, L has a formula expressing the following on frames:
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The logic of monadic Boolean algebras (aka $\mathrm{S5}$) is one of the simplest examples of locally finite modal logics.

However, the logic $\mathrm{S5} * \mathrm{S5}$ with two monadic operators is not locally finite; moreover, its one-variable fragment is infinite.

Indeed, $\mathrm{S5} * \mathrm{S5}$ is the logic of frames $(W, \sim_1, \sim_2)$ with two equivalence relations. So (1) does not hold for $\mathrm{S5} * \mathrm{S5}$.

We need axioms that make all $\Diamond \in \mathcal{A}$ dependant.

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[N. Bezhanishvili, 2002] Every extension of

$$\mathrm{S5} * \mathrm{S5} + \Diamond_1\Diamond_2 p \leftrightarrow \Diamond_2\Diamond_1 p$$

is locally finite.

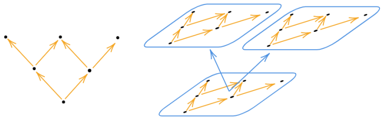
Part II. Conditions for local finiteness in the polymodal case.

Let $\mathcal{A}$ and $\mathcal{B}$ be disjoint alphabets of modalities.

Definition

Let $\mathsf{I} = (I, (S_a)_{a\in\mathcal{A}})$ be an $\mathcal{A}$-frame, $(\mathsf{F}_i)_{i\in I}$ be $\mathcal{B}$-frames, $\mathsf{F}_i = (X_i, (R_{i,b})_{b\in\mathcal{B}})$. The *lexicographic sum* $\sum^{\mathrm{lex}}_{\mathsf{I}} \mathsf{F}_i$ is the $(\mathcal{A}\cup\mathcal{B})$-frame $\left(\bigsqcup_{i\in I} X_i, (S^{\mathrm{lex}}_a)_{a\in\mathcal{A}}, (R_b)_{b\in\mathcal{B}}\right)$:

$$
\begin{array}{rcl}
(i,w)S^{\mathrm{lex}}_a(j,u) & \text{iff} & iSj, \\
(i,w)R_b(j,u) & \text{iff} & i = j \;\&\; wR_{i,b}u.
\end{array}
$$



In many cases, the modal logic of a class of sums inherits "good" properties of the logics of summands/indices.

Transfer results for sums in modal logic:

Axiomatization [Kracht 1993; Beklemishev 2007; Balbiani 2009; Balbiani & Mikulás 2013; Balbiani & Sh, 2014; Balbiani & Fernández-Duque 2016]

Finite model property and decidability [Babenyshev & Rybakov 2010; Sh 2018]

Computational complexity [Sh 2008; Sh 2020]

Local finiteness [this talk]

Semantic

Let L_1 and L_2 be logics in disjoint alphabets of modalities $\mathcal{A}$ and $\mathcal{B}$, respectively.

$\sum^{\text{lex}}_{L_1} L_2$ is the logic of lexicographic sums of their frames.

Theorem (Main result)

If L_1 and L_2 are locally finite, then $\sum^{\text{lex}}_{L_1} L_2$ is locally finite.

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Syntactic

Let $\Phi(\mathcal{A}, \mathcal{B})$ the set of all formulas

$\Diamond_b \Diamond_a p \to \Diamond_a p, \; \Diamond_a \Diamond_b p \to \Diamond_a p, \; \Diamond_a p \to \Box_b \Diamond_a p$

with $\Diamond_a$ in $\mathcal{A}$ and $\Diamond_b$ in $\mathcal{B}$.

$L_1 \oplus L_2$ is the smallest logic that contains

$$L_1 \cup L_2 \cup \Phi(\mathcal{A}, \mathcal{B})$$

Theorem

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Semantic $\Rightarrow$ Syntactic:

[Balbiani and others] In many cases, $\sum^{\text{lex}}_{L_1} L_2 = L_1 \oplus L_2$.

Observation. For all logics, $\sum^{\text{lex}}_{L_1} L_2 \subseteq L_1 \oplus L_2$ provided that $L_1 \oplus L_2$ is Kripke complete.

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Corollary

Recall: The least transitive logic $\mathrm{K4}$ is given by the axiom $\Diamond\Diamond p \to \Diamond p$.
Formulas of *finite height*: $B_0 = \bot, \ B_{i+1} = p_{i+1} \to \Box(\Diamond p_{i+1} \vee B_i)$.

[Segerberg 1971; Maksimova 1975] Let $L \supseteq \mathrm{K4}$. Then

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Corollary. Let $L \supseteq \mathrm{K4}(\Diamond_1) \oplus \mathrm{K4}(\Diamond_2)$. Then

$$L \text{ is locally finite iff } L \vdash B_m(\Diamond_1) \wedge B_n(\Diamond_2) \text{ for some } n, m.$$

Auxiliary step in the proof of main result

For a frame $F = (W, (R_\Diamond)_{\Diamond \in \mathcal{A}})$, let F^{r} be the frame $(W, (R^{\mathrm{r}}_\Diamond)_{\Diamond \in \mathcal{A}})$, where R^{r}_a is the reflexive closure of $R_\Diamond$. For a class $\mathcal{F}$ of frames, $\mathcal{F}^{\mathrm{r}} = \{F^{\mathrm{r}} \mid F \in \mathcal{F}\}$.

Expectable theorem.

Let $\mathcal{F}$ be a class of frames. The $\mathrm{Log}(\mathcal{F})$ is locally finite iff $\mathrm{Log}(\mathcal{F}^{\mathrm{r}})$ is locally finite.

"Only if" is trivial. "If" is based on the following lemma (with unexpectedly convoluted proof)

Lemma. Let F be an irreflexive $\mathcal{A}$-frame. Assume that the logic of the frame F is locally finite. Then for every $k < \omega$, every k-generated subalgebra of $\mathrm{Alg}(F)$ is contained in a $(k + 3|\mathcal{A}|)$-generated subalgebra of $\mathrm{Alg}(F^{\mathrm{r}})$.

Question. Should we expect that Expectable theorem holds for locally finite algebras (not logics/varieties)?

What could be other operations on frames that preserve good properties of their algebras/logics?

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Example

Let $(\omega^n, \preceq)$ be the n-th direct power of $(\omega, \leq)$.
[2019] For all finite n, $\mathrm{Alg}(\omega^n, \preceq)$ is locally finite.

$n = 1$: an exercise; $n > 1$: not that easy.

Assume that algebras (logics) of frames F_1 and F_2 are locally finite. Is the algebra (logic) of the direct product $\mathsf{F}_1 \times \mathsf{F}_2$ locally finite?
For quasi-orders? Partial orders? For well-founded orders? At least, for well-orders is should be true...

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Thank you!