Medvedev's logic and products of converse well orders

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This talk is about modal logics of *Noetherian* (in other terms, *converse well-founded*) posets which are substructures of direct products of converse well-ordered sets, in particular, the products without an upper part.

Given frames $\mathfrak{F}_i = (W_i, R_i)$, $i \in I$, their *direct product* is the frame $\prod_I \mathfrak{F}_i = (W, R)$ where $W = \prod_{i \in I} W_i$, the Cartesian product of the sets W_i, and R is defined point-wise: xRy iff $x_i R_i y_i$ for all $i \in I$. Given a frame $\mathfrak{F}$, we write $\mathfrak{F}^n$ for its nth direct power.

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Two examples: $\mathrm{Grz.2}$ and modal Medvedev's logic

1. $\mathrm{Grz.2} = \mathrm{Grz} + \Diamond\Box p \to \Box\Diamond p$, Grzegorczyk logic extended with the axiom of weak directedness.

   Theorem (Maksimova, Shehtman, Skvortsov, 1979).

   $$\mathrm{Grz.2} = Log\{(2, \geq)^n \mid n < \omega\}$$

2. Cubes $(2, \geq)^n$ without the top element are called *Medvedev's frames*. *Modal Medvedev's logic* Mdv is the modal logic of these structures:

   $$\mathrm{Mdv} = \mathrm{Log}\{(2, \geq)^n \setminus \{\mathrm{top}\} : n < \omega\}$$

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In spite of the similarity in the above semantic characterizations, logical properties of $\mathrm{Grz.2}$ and Mdv are different. In particular:
$\mathrm{Grz.2}$ is given by a finite set of axioms, so in view of its completeness with respect to a class of finite frames, it is decidable.
It is well known that both modal and intuitionistic Medvedev logics are not finitely axiomatizable [Maksimova, Shehtman, Skvortsov 1979; Prucnal 1979].

Whether the Medvedev logic (modal or intuitionistic) is recursively axiomatizable is an old-standing open problem.

$$\Gamma(n) := Log((\omega, \geq)^n),$$
$$\Delta(n) := Log((\omega, \geq)^n \setminus \{\mathrm{top}\}).$$

$\Delta(1) = \Gamma(1) = \mathrm{Grz.3}$, the logic of all (finite) linear Noetherian non-strict partial orders.

All $\Gamma(n)$ and $\Delta(n)$ have the fmp (trivially):

$$\Gamma(n) = \mathrm{Log}\{(m, \geq)^n : m < \omega\},$$
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Theorem. For all finite n, we have:

$$\Gamma(n) = \mathrm{Log}\{\prod_{i<n}(\alpha_i, \geq) : \alpha_i \in \mathrm{Ord}\},$$
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Proof via selective filtration and Dickson's lemma.

Corollary.
For all finite n, $\mathrm{Log}(\mathfrak{F}) = \mathrm{Grz.3}$ implies $\mathrm{Log}(\mathfrak{F}^n) = \Gamma(n)$ and $\mathrm{Log}(\mathfrak{F}^n \setminus \{\mathrm{top}\}) = \Delta(n)$.

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$\mathrm{Grz.2}$ is the logic of all (finite) Noetherian non-strict posets with a greatest element.

Recall that $\mathrm{Grz.2} = Log\{(2, \geq)^n \mid n < \omega\}$.

Corollary. $\bigcap_{n<\omega} \Gamma(n) = \mathrm{Grz.2}$.

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Mdv, $\Gamma(n)$, $\Delta(n)$ are co-recursively enumerable.

Recall that:

Mdv is not finitely axiomatizable;
whether Mdv is RE is an old open problem.

Q. Are the logics $\Gamma(n)$, $\Delta(n)$, $2 \leq n < \omega$, finitely axiomatizable? RE?

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Theorem (Shehtman, 1990). $\mathrm{Mdv.2} = \mathrm{Grz.2}$

Theorem. $\Delta(n).2 = \Gamma(n)$ for all $n < \omega$.

Corollary. If $\Delta(n)$ is finitely (recursively) axiomatizable, then so is $\Gamma(n)$.

Inclusions between logics $\Gamma(n)$ and $\Delta(n)$:

$$\begin{tikzcd}[column sep=small]
 & \Delta(1) = \Gamma(1) = \mathrm{Grz.3} & & & & \\
 & \Gamma(2) \arrow[u] & & & & \\
\Delta(2) \arrow[ur] & & \Gamma(3) \arrow[ul] & & & \\
 & \Delta(3) \arrow[ul] \arrow[ur] & & \Gamma(4) \arrow[ul] & & \\
 & & \Delta(4) \arrow[ul] \arrow[ur] & & \ddots \arrow[ul] & \\
 & & & \ddots \arrow[ul] & & \mathrm{Grz.2} \arrow[ul] \\
 & & & & \mathrm{Mdv} \arrow[ul] \arrow[ur] & \\
 & & & & \mathrm{Grz} \arrow[u] &
\end{tikzcd}$$

Proposition For all $n < \omega$,

1. $\Delta(1) = \Gamma(1) = \mathrm{Grz.3}$,
2. $\mathrm{Grz} \subset \Delta(n) \subset \Gamma(n)$ if $n \geq 2$,
3. $\Delta(n+1) \subset \Delta(n)$ and $\Gamma(n+1) \subset \Gamma(n)$,
4. $\Delta(n) \nsubseteq \Gamma(n+1)$ and $\Gamma(n) \nsubseteq \Delta(2)$.

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1. n-adic fragments of SO logic of ω.

By the standard translation argument, the logics $\Gamma(n)$ and $\Delta(n)$ are fragments of the *n-adic* (i.e., relation variables are n-ary) second order logic over natural numbers with the standard ordering. Propositional variables p are interpreted as n-ary predicates on ω, the order on tuples in $(\omega, \geq)^n$ is interpreted via conjunctions $\bigwedge_{i<n} x_i \geq y_i$.

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2. Interval Temporal Logic.

The logics $\Gamma(2)$ and $\Delta(2)$ can be considered in the context of interval temporal logic. Let W be the set of closed segments $[m, n]$ of integer numbers containing a fixed integer (e.g., 0):

$$W := \{[m, n] : m \leq 0 \leq n\}.$$

It is immediate that $\mathrm{Log}(W, \supseteq)$ is $\Gamma(2)$ and $\mathrm{Log}(W \setminus \{[0, 0]\}, \supseteq)$ is $\Delta(2)$ according to the fact that $(W, \supseteq)$ is isomorphic to $(\omega, \geq)^2$; the isomorphism is defined by letting

$$(m, n) \mapsto [-m, n].$$

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3. Modal logics of model-theoretic relations

Given a class $\mathcal{C}$ of models, a binary relation $\mathcal{R}$ between models, and a model-theoretic language L, one can consider the *modal logic* $\mathrm{ML}^L(\mathcal{C}, \mathcal{R})$, where variables are evaluated by sentences of L and the modal operator is interpreted via $\mathcal{R}$, and

$\mathfrak{A} \models \Diamond\varphi$ *("φ is possible at $\mathfrak{A}$") iff*
$\mathfrak{B} \models \varphi$ *for some* $\mathfrak{B}$ *with* $\mathfrak{A}\,\mathcal{R}\,\mathfrak{B}$

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During last decades, modal logics of various relations between models and theories of arithmetics [Shavrukov, Visser, Berarducci, Ignatiev, Hamkins, and others] and between models of set theory [Hamkins, Löwe, Block, Leibman, Tanmay, and others] were considered.

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In general, $\mathrm{ML}^L(\mathcal{C}, \mathcal{R})$ depends on the model-theoretic language L.

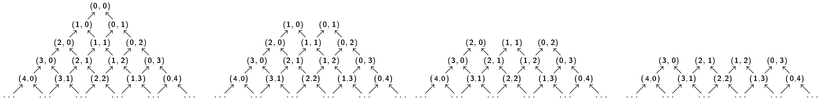
Observation: for two L and K, $L \subseteq K$ implies $\mathrm{ML}^L(\mathcal{C}, \mathcal{R}) \supseteq \mathrm{ML}^K(\mathcal{C}, \mathcal{R})$.
$\mathrm{MTh}^L(\mathcal{C}, \mathcal{R})$ is *robust* iff for every language $K \supseteq L$ we have

$$\mathrm{MTh}^K(\mathcal{C}, \mathcal{R}) = \mathrm{MTh}^L(\mathcal{C}, \mathcal{R}).$$

Intuitively, the robust theory can be considered as a "true" modal logic of the model-theoretic relation $\mathcal{R}$ on $\mathcal{C}$.

Theorem. Let $n < \omega$, and let τ be a signature consisting of n unary predicates and possibly some constants. The robust modal logic of the class of models of τ with the submodel relation is $\Gamma(2^n)$ if τ has at least one constant, and $\Delta(2^n)$ otherwise.

For an ordinal α, the α*th level* of a Noetherian frame $\mathfrak{F} = (X, \geq)$ consists of all points of $\mathfrak{F}$ having the rank α in the well-founded frame $(X, \leq)$.
Let $\mathfrak{P}_{n,m}$ denote the restriction of the frame $(\omega, \geq)^n$ to its levels $\geq m$



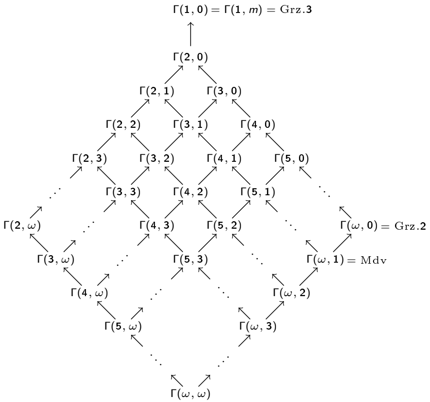
$\mathfrak{P}_{2,0}$ $\mathfrak{P}_{2,1}$ $\mathfrak{P}_{2,2}$ $\mathfrak{P}_{2,3}$

For $1 \leq n < \omega$, $0 \leq m < \omega$, let $\Gamma(n, m) := \mathrm{Log}(\mathfrak{P}_{n,m})$;
$\Gamma(\omega, m) := \mathrm{Log}\{\mathfrak{P}_{n,m} : n < \omega\}$, $\Gamma(n, \omega) := \mathrm{Log}\{\mathfrak{P}_{n,m} : m < \omega\}$,
$\Gamma(\omega, \omega) := \mathrm{Log}\{\mathfrak{P}_{n,m} : m, n < \omega\}$.
Hence:

(i) $\Gamma(1, m) = \mathrm{Grz.3}$ for all $m \leq \omega$,

(ii) $\Gamma(n, 0) = \Gamma(n)$ and $\Gamma(n, 1) = \Delta(n)$,

(iii) $\Gamma(\omega, 0) = \mathrm{Grz.2}$ and $\Gamma(\omega, 1) = \mathrm{Mdv}$.

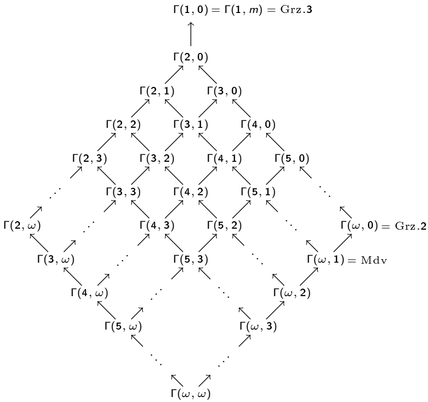


Theorem

1. $\Gamma(n,m) \supseteq \Gamma(n',m')$ if $n \leq n' \leq \omega$ and $m \leq m' \leq \omega$,
2. $\Gamma(n,\omega) \not\subseteq \Gamma(n+1,0)$,
3. $\Gamma(n,m) \not\subseteq \Gamma(n',m')$ if $\binom{m+n-1}{n-1} < \binom{m'+n'-1}{n'-1}$.

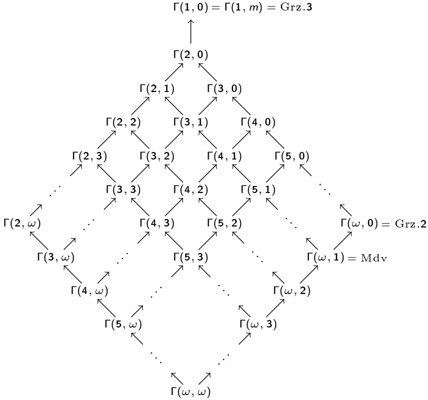
Corollary. $\Gamma(n,m) \supset \Gamma(n',m)$ if $n < n'$; $\Gamma(n,m) \supset \Gamma(n,m')$ if $m < m'$.

Q. What about the inclusions of the logics that are not under the scope of the Theorem? E.g., is $\Gamma(3,1) \subseteq \Gamma(2,2)$?



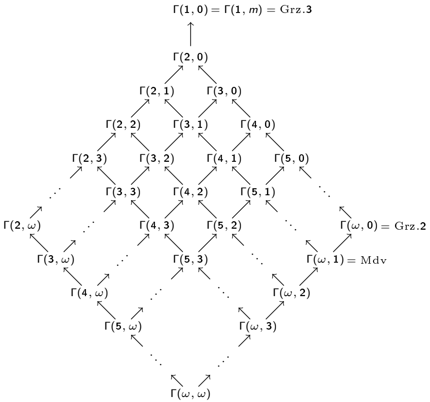
Recall that: $\mathrm{Grz.2} = \mathit{Log}\{(2, \geq)^n \mid n < \omega\}$, $\quad \mathrm{Mdv} = \mathrm{Log}\{(2, \geq)^n \setminus \{\mathrm{top}\} : n < \omega\}$

Equivalently, $\mathrm{Grz.2} = \mathrm{Log}(P_\omega(\omega), \supseteq)$, $\quad \mathrm{Mdv} = \mathrm{Log}(P_\omega(\omega) \setminus \{\emptyset\}, \supseteq)$



Recall that: $\mathrm{Grz.2} = Log\{(2, \geq)^n \mid n < \omega\}, \quad \mathrm{Mdv} = \mathrm{Log}\{(2, \geq)^n \setminus \{\mathrm{top}\} : n < \omega\}$

Equivalently, $\mathrm{Grz.2} = \mathrm{Log}(P_\omega(\omega), \supseteq), \quad \mathrm{Mdv} = \mathrm{Log}(P_\omega(\omega) \setminus \{\emptyset\}, \supseteq)$

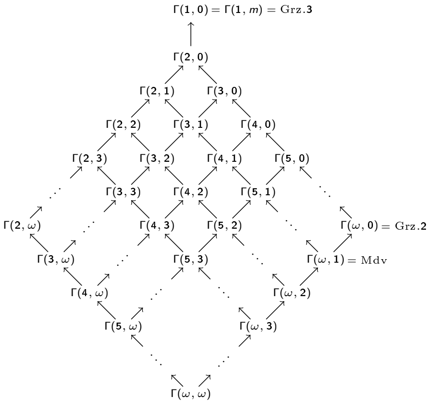


Given cardinals $\kappa < \lambda$, let $\mathcal{P}_{\lambda,\kappa}(X) := \mathcal{P}_\lambda(X) \setminus \mathcal{P}_\kappa(X) = \{A \subseteq X : \kappa \leq |A| < \lambda\}$.

Proposition. For all $m < \omega$, $\Gamma(\omega, m) = \mathrm{Log}(\mathcal{P}_{\omega,m}(\omega), \supseteq) = \mathrm{Log}\{(\mathcal{P}_{n+1,m}(n), \supseteq) : n < \omega\}$.

Related systems, the intuitionistic logics of $(\mathcal{P}(\omega) \setminus \mathcal{P}(m), \supseteq)$, $0 < m < \omega$, were considered in [Shehtman, Skvortsov, 1986]: none of them are finitely axiomatizable.

We conjecture that the logics $\Gamma(\omega, m)$ are not finitely axiomatizable as well.


Γ(1, 0) = Γ(1, m) = Grz.3
Γ(2, 0)
Γ(2, 1)
Γ(3, 0)
Γ(2, 2)
Γ(3, 1)
Γ(4, 0)
Γ(2, 3)
Γ(3, 2)
Γ(4, 1)
Γ(5, 0)
Γ(3, 3)
Γ(4, 2)
Γ(5, 1)
Γ(2, ω)
Γ(4, 3)
Γ(5, 2)
Γ(ω, 0) = Grz.2
Γ(3, ω)
Γ(5, 3)
Γ(ω, 1) = Mdv
Γ(4, ω)
Γ(ω, 2)
Γ(5, ω)
Γ(ω, 3)
Γ(ω, ω)

Thank you!