Modal logics of finite direct powers of ω have the finite model property

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$$(x_1, \dots, x_n) \preceq (y_1, \dots, y_n) \quad \Leftrightarrow \quad \sum_{i=1}^{n-1} (y_i - x_i)^2 \leq (x_n - y_n)^2 \ \&\ x_n \leq y_n,$$

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This talk:

The logics of direct powers of $(\omega, \leq)$ and $(\omega, <)$.

This talk is about
modal logics of direct products of Kripke frames,
their finite model property,
and local finiteness.

Language: a countable set VAR (propositional variables), Boolean connectives, a unary connective $\Diamond$ ($\Box$ abbreviates $\neg\Diamond\neg$).

Normal modal logics: Definition 1

A set of modal formulas L is a *normal modal logic* if L contains

- all tautologies
- $\Diamond\bot \leftrightarrow \bot, \quad \Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$

and is closed under MP, Sub, and Mon: if $(\varphi \to \psi) \in \mathrm{L}$, then $(\Diamond\varphi \to \Diamond\psi) \in \mathrm{L}$.

Normal modal logics: Definition 2

A modal algebra is a BA endowed with a unary operation that distributes over finite disjunctions.

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Kripke semantics

A *(Kripke) frame* F is a pair (W, R), where $W \neq \varnothing$, $R \subseteq W \times W$.
A *model* M on F is a pair (F, θ) where $\theta : \mathrm{VAR} \to \mathcal{P}(W)$.

$M, x \vDash p$ iff $x \in \theta(p)$, $\quad M, x \vDash \Diamond\varphi$ iff $M, y \vDash \varphi$ for some y with xRy.

Log$(F) = \{\varphi \mid F \vDash \varphi\}$, where $M, x \vDash \varphi$ for every M on F and every x in M.

The *algebra Alg(F) of a frame* $F = (W, R)$ is the modal algebra $(\mathcal{P}(W), R^{-1})$.

Hence: $F \vDash \varphi$ iff $Alg(F) \vDash \varphi = \top$.

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A logic L has the *finite model property* if L is the logic of a class $\mathcal{C}$ of finite frames/algebras: $\mathrm{L} = \bigcap\{Log(F) \mid F \in \mathcal{C}\}$.

An algebra A is *locally finite* if every finitely generated subalgebra of A is finite.

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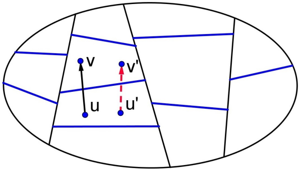
$\not\Leftarrow$

Let $F = (W, R)$ be a frame. A partition $\mathcal{A}$ of W is *tuned* if for every $U, V \in \mathcal{A}$,

$$\exists u \in U\, \exists v \in V\ uRv \quad \Rightarrow \quad \forall u \in U\, \exists v \in V\ uRv.$$

F is said to be *tunable* if every finite partition $\mathcal{A}$ of F admits a finite tuned refinement.



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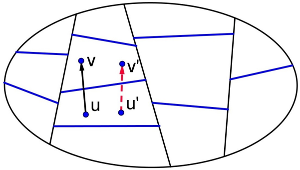

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TFAE:

- $\mathcal{A}$ is tuned in F
- The equivalence $\sim$ defined by $\mathcal{A} = W/\sim$ satisfies the condition

$$\sim \circ R \subseteq R \circ \sim,$$

  i.e., $\sim$ is a bisimulation w.r.t. R on W.
- $x \mapsto [x]_{\mathcal{A}}$ if a p-morphism from F onto the "Franzen's filtration" $(\mathcal{A}, R_{\mathcal{A}})$, where for $U, V \in \mathcal{A}$,

$$UR_{\mathcal{A}}V \text{ iff } \exists u \in U\, \exists v \in V\ uRv$$

  [Segerberg, K.: Franzen's proof of Bull's theorem. Ajatus 35, 216–221 (1973)]
- Finite unions of elements of $\mathcal{A}$ form a subalgebra of $Alg(F)$.

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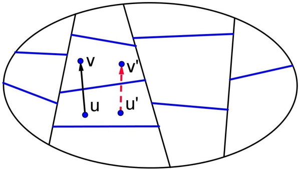

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Example: The logics/algebras of $(\omega, \leq)$ and $(\omega, <)$

Bull, 1965; Schindler, 1970/Segerberg, 1970: $Log(\omega, \leq)$ and $Log(\omega, <)$ have the fmp.

Moreover, the algebras $Alg(\omega, \leq)$ and $Alg(\omega, <)$ are locally finite, since $(\omega, \leq)$ and $(\omega, <)$ are (easily!) tunable:

refine a given finite partition $\mathcal{A}$ of ω in such a way that

- all elements of the refinement $\mathcal{B}$ are infinite or singletons, and
- singletons form an initial segment of ω.

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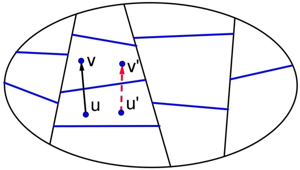

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A logic L is *locally finite* (or *locally tabular*) if for all $k < \omega$ there are only finitely many k-formulas (i.e., formulas in k variables) up to $\leftrightarrow_{\mathrm{L}}$.

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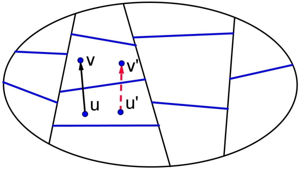

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TFAE:

L is locally finite.	Every finitely generated Lindenbaum-Tarski (i.e., free) algebra of L is finite.	The variety of L-algebras is *locally finite*, i.e., every finitely generated L-algebra is finite.

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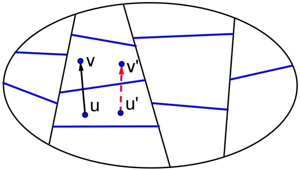

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Malcev, 1970s: The variety $\mathrm{Var}(A)$ of a finite signature is LF iff there exists $f : \omega \to \omega$ s.t. the cardinality of a subalgebra of A generated by $m < \omega$ elements is $\leq f(m)$.

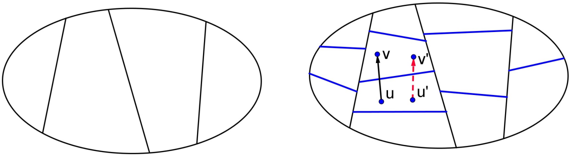
Shehtman & Sh, 2016: $Log(F)$ is LF iff there exists $f : \omega \to \omega$ s.t. every finite partition $\mathcal{A}$ of F admits a tuned finite refinement $\mathcal{B}$ with $|\mathcal{B}| \leq f(|\mathcal{A}|)$.

$Alg(\omega, \leq)$ and $Alg(\omega, <)$ are locally finite.

Segerberg, 1971; Maksimova, 1975:
The logic of a transitive frame F is locally finite iff F is of finite height.

$$Log(F) \text{ is LF} \quad \begin{matrix} \Rightarrow \\ \not\Leftarrow \end{matrix} \quad Alg(F) \text{ is LF} \quad \begin{matrix} \Rightarrow \\ \not\Leftarrow \end{matrix} \quad Log(F) \text{ has the FMP}$$

$Alg(F)$ is locally finite iff F is tunable



Example: $Alg(\omega, \leq)$ and $Alg(\omega, <)$ are locally finite

Proof. The frames $(\omega, \leq)$ and $(\omega, <)$ are tunable (easy).

$(\omega^n, \preceq)$ is the n-th direct power of $(\omega, \leq)$, i.e., for $x, y \in \omega^n$
$x \preceq y$ iff $x(i) \leq y(i)$ for all $i < n$.

$(\omega^n, \prec)$ is the direct power $(\omega, <)^n$:
$x \prec y$ iff $x(i) < y(i)$ for all $i < n$.

Theorem

For all finite $n > 0$, $Alg(\omega^n, \preceq)$ and $Alg(\omega^n, \prec)$ are locally finite.

Corollary

The logics $Log(\omega^n, \preceq)$ and $Log(\omega^n, \prec)$ have the finite model property.

Induction step

Consider a non-empty $V \subseteq \omega^n$. Put

$$\begin{aligned} J(V) &= \{i < n \mid \exists x \in V\, \exists y \in V\; x(i) \neq y(i)\}, \\ I(V) &= \{i < n \mid \forall x \in V\, \forall y \in V\; x(i) = y(i)\} = n \setminus J(V). \end{aligned}$$

The *hull of* V is the set

$$\overline{V} = \{y \in \omega^n \mid \forall i \in I(V)\, (y(i) = x(i) \text{ for some (for all) } x \in V)\}.$$

V is *pre-cofinal* if it is cofinal in its hull, i.e.,

$$\forall x \in \overline{V}\, \exists y \in V\; x \preceq y.$$

A partition $\mathcal{A}$ of $V \subseteq \omega^n$ is *monotone* if

- all of its elements are pre-cofinal, and
- for all $x, y \in V$ such that $x \preceq y$ we have $J([x]_{\mathcal{A}}) \subseteq J([y]_{\mathcal{A}})$,

where $[x]_{\mathcal{A}}$ is the element of $\mathcal{A}$ containing x.

If $\mathcal{A}$ is a monotone partition of ω^n, then $\mathcal{A}$ is tuned in $(\omega^n, \preceq)$ and in $(\omega^n, \prec)$. Let $A, B \in \mathcal{A}$, $x, y \in A$, $x \preceq z \in B$. Let u be the following point in ω^n:

$$u(i) = y(i) + 1 \text{ for } i \in J(A), \text{ and } u(i) = z(i) \text{ for } i \in I(A). \tag{2}$$

We have

$$\{i < n \mid u(i) \neq z(i)\} \subseteq n \setminus I(A) = J(A) \subseteq J(B);$$

the first inclusion follows from (2), the second follows from the monotonicity of $\mathcal{A}$. Hence, we have $u(i) = z(i)$ for all $i \in I(B)$. By the definition of $\overline{B}$, we have $u \in \overline{B}$. Since B is cofinal in $\overline{B}$ (we use monotonicity again), for some $u' \in B$ we have $u \preceq u'$.

By (2), we have $y(i) \leq u(i)$ for all $i < n$: indeed, $y(i) = x(i) \leq z(i) = u(i)$ for $i \in I(A)$, and $u(i) = y(i) + 1$ otherwise. Thus, $y \preceq u$, and so $y \preceq u'$. It follows that $\mathcal{A}$ is tuned in $(\omega^n, \preceq)$.

In order to show that $\mathcal{A}$ is tuned in $(\omega^n, \prec)$, we now assume that $x \prec z$. Then we have $y(i) < u(i)$ for all $i < n$, since $y(i) = x(i) < z(i) = u(i)$ for $i \in I(A)$, and $u(i) = y(i) + 1$ otherwise. Hence $y \prec u$. Since $u \preceq u'$, we have $y \prec u'$, as required.

Let $\mathcal{A}$ be a finite partition of ω^n.

For $k \in \omega$ let $U_k = \{y \in \omega^n \mid y(i) \geq k \text{ for all } i < n\}$. Since $\mathcal{A}$ is finite, we can choose a natural number k_0 such that

$$\text{if } y \in U_{k_0}, \text{ then } [y]_{\mathcal{A}} \text{ is cofinal in } \omega^n. \tag{3}$$

Indeed, if $A \in \mathcal{A}$ is not cofinal in ω^n, then $U_{k_A} \cap A = \varnothing$ for some $k_A < \omega$; hence, (3) holds whenever k_0 is greater than every such k_A.

It follows that the partition $\mathcal{A}{\restriction}U_{k_0}$ is monotone: it consists of sets that are cofinal in ω^n (and so, they are obviously pre-cofinal), and $J(A) = n$ for all $A \in \mathcal{A}{\restriction}U_{k_0}$.

We are going to extend this partition step by step in order to obtain a sequence of finite monotone partitions of $U_{k_0-1}, \ldots, U_0 = \omega^n$, respectively refining $\mathcal{A}{\restriction}U_{k_0-1}, \ldots, \mathcal{A}{\restriction}U_0 = \mathcal{A}$.

First, let us describe the construction for the case $k_0 = 1$, the crucial technical step of the proof.

Claim A. Suppose that $\mathcal{B}$ is a finite monotone partition of U_1 refining $\mathcal{A}{\restriction}U_1$. Then there exists a finite monotone partition $\mathcal{C}$ of ω^n refining $\mathcal{A}$ such that $\mathcal{B} \subseteq \mathcal{C}$. $\mathcal{C}$ will be the union of $\mathcal{B}$ and a partition of the set

$$V = \{x \in \omega^n \mid x(i) = 0 \text{ for some } i < n\} = \omega^n \setminus U_1.$$

To construct the required partition of V, for $I \subseteq n$ put

$$V_I = \{x \mid \forall i < n\, (i \in I \Leftrightarrow x(i) = 0)\}.$$

Then $\{V_I \mid \varnothing \neq I \subseteq n\}$ is a partition of V, $V_\varnothing = U_1$.

Each V_I considered with the order $\preceq$ on it is isomorphic to $(\omega^{n-|I|}, \preceq)$. Thus, by the induction hypothesis, for a non-empty $I \subseteq n$ we have:

$$\text{Each finite partition of } V_I \text{ admits a finite monotone refinement.} \tag{4}$$

For $I \subseteq n$, by induction on the cardinality of I we define a finite partition $\mathcal{C}_I$ of V_I.

We put $\mathcal{C}_\varnothing = \mathcal{B}$.

Assume that I is non-empty. Consider the projection $\mathrm{Pr}_I : x \mapsto y$ such that $y(i) = 0$ whenever $i \in I$, and $y(i) = x(i)$ otherwise. Note that for all $K \subset I$, $x \in V_K$ implies $\mathrm{Pr}_I(x) \in V_I$. Let $\mathcal{D}$ be the partition induced on V_I by the family

$$\mathcal{A} \cup \bigcup_{K \subset I} \{\mathrm{Pr}_I(A) \mid A \in V_K\}. \tag{5}$$

By an immediate induction argument, $\mathcal{D}$ is finite. Let $\mathcal{C}_I$ be a finite monotone refinement of $\mathcal{D}$, which exists according to (4).

We put

$$\mathcal{C} = \bigcup_{I \subseteq n} \mathcal{C}_I.$$

Then $\mathcal{C}$ is a finite refinement of $\mathcal{A}$. We have to check monotonicity.

Every element A of $\mathcal{C}$ is pre-cofinal, because A is an element of a monotone partition $\mathcal{C}_I$ for some I. In order to check the second condition of monotonicity, we consider x, y in ω^n with $x \preceq y$ and show that

$$J([x]_{\mathcal{C}}) \subseteq J([y]_{\mathcal{C}}). \tag{6}$$

Let $x \in V_I$, $y \in V_K$ for some $I, K \subseteq n$. Since $x \preceq y$, we have $K \subseteq I$. If $K = I$, then (6) holds, since in this case $[x]_{\mathcal{C}}$ and $[y]_{\mathcal{C}}$ belong to the same monotone partition $\mathcal{C}_I$. Assume that $K \subset I$. In this case we have:

$$J([x]_{\mathcal{C}}) \subseteq J([\mathrm{Pr}_I(y)]_{\mathcal{C}}) \subseteq J(\mathrm{Pr}_I([y]_{\mathcal{C}})) \subseteq J([y]_{\mathcal{C}}).$$

To check the first inclusion, we observe that $\mathrm{Pr}_I(y)$ belongs to V_I (since $K \subset I$). This means that $[x]_{\mathcal{C}}$ and $[\mathrm{Pr}_I(y)]_{\mathcal{C}}$ are elements of the same partition $\mathcal{C}_I$. We have $x \preceq \mathrm{Pr}_I(y)$, since $x \in V_I$ and $x \preceq y$. Now the first inclusion follows from monotonicity of $\mathcal{C}_I$. By (5), $\mathrm{Pr}_I([y]_{\mathcal{C}})$ is the union of some elements of $\mathcal{C}_I$ (since $K \subset I$ and $[y]_{\mathcal{C}} \in \mathcal{C}_K$); trivially, $\mathrm{Pr}_I(y) \in \mathrm{Pr}_I([y]_{\mathcal{C}})$, hence $[\mathrm{Pr}_I(y)]_{\mathcal{C}}$ is a subset of $\mathrm{Pr}_I([y]_{\mathcal{C}})$. This yields the second inclusion. The third inclusion is immediate from the definition. Thus, we have (6), which proves the claim.

From Claim A it is not difficult to obtain the following:

Claim B. Let $0 < k < \omega$. If $\mathcal{B}$ is a finite monotone partition of U_k refining $\mathcal{A}{\restriction}U_k$, then there exists a finite monotone partition $\mathcal{C}$ of U_{k-1} refining $\mathcal{A}{\restriction}U_{k-1}$ such that $\mathcal{B} \subseteq \mathcal{C}$. Consider the translation $\mathrm{Tr} : U_{k-1} \to \omega^n$ taking $(x_i)_{i<n}$ to $(x_i - k + 1)_{i<n}$. Let $\mathcal{B}'$ be the set $\{\mathrm{Tr}(A) \mid A \in \mathcal{B}\}$ of images of elements of $\mathcal{B}$ by Tr, and $\mathcal{A}'$ be the set $\{\mathrm{Tr}(A) \mid A \in \mathcal{A}{\restriction}U_{k-1}\}$. Then $\mathcal{A}'$ is a partition of ω^n, $\mathcal{B}'$ is a finite monotone partition of U_1 refining $\mathcal{A}'{\restriction}U_1$. By Claim A, there exists a finite monotone partition $\mathcal{C}'$ of ω^n refining $\mathcal{A}'$ such that $\mathcal{B}' \subseteq \mathcal{C}'$. The family $\mathcal{C} = \{\mathrm{Tr}^{-1}(A) \mid A \in \mathcal{C}'\}$ is the required partition of U_{k-1}.

Applying Claim B k_0 times, we obtain the required refinement of $\mathcal{A}$.

Question

Let frames F_1 and F_2 be tunable. Is the direct product $F_1 \times F_2$ tunable?

In the other words:
if $Alg(F_1)$ and $Alg(F_2)$ are LF, is the algebra $Alg(F_1 \times F_2)$ LF?

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Proposition. For every ordinal $\alpha > 0$, the algebras $Alg(\alpha, \leq)$, $Alg(\alpha, <)$ are LF.

Proof. The frames $(\alpha, \leq)$, $(\alpha, <)$ are tunable (an easy induction on α).

Conjecture

If $(\alpha_i)_{i<n}$ is a finite family of ordinals, then the algebras of the direct products $\prod_{i<n}(\alpha_i, \leq)$, $\prod_{i<n}(\alpha_i, <)$ are locally finite.

Question

Let $n > 1$. Are logics $Log(\omega^n, \preceq)$ and $Log(\omega^n, \prec)$ decidable or at least recursively axiomatizable?

In the one-dimensional case, decidability is a classical result (Bull, 65; Schindler, 1970; Segerberg, 1970).

Every extension of a locally finite logic is locally finite, and so has the finite model property.

The algebras of the frames $(\omega^n, \preceq)$ and $(\omega^n, \prec)$ are locally finite, the logics of these frames are not (since these frames are of infinite height).

Bull, 1965: Every extension of $Log(\omega, \leq)$ has the finite model property.

Question

Let L be an extension of $Log(\omega^n, \preceq)$ for some finite $n > 1$. Does L have the finite model property?

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A *subframe* of (W, R) is the restriction $(V, R \cap (V \times V))$, where $V \neq \varnothing$.

Proposition. If the algebra of a frame F is locally finite, then the algebra of any subframe of F is also locally finite.

Proof. Given a partition $\mathcal{A}$ of a subframe G, consider the partition $\mathcal{A} \cup \{W \setminus V\}$ of F. Tune it, and collect elements that are subsets of G.

Corollary

For all finite n, if F is a subframe of $(\omega^n, \preceq)$ or of $(\omega^n, \prec)$, then $Alg(F)$ is locally finite, and $Log(F)$ has the finite model property.

(Glivenko, 1929) $\mathrm{CL} \vdash \varphi$ iff $\mathrm{IL} \vdash \neg\neg\varphi$
(Matsumoto, 1955) $\mathrm{S5} \vdash \varphi$ iff $\mathrm{S4} \vdash \neg\Box\neg\Box\varphi$

In Kripke semantics:

IL is the logic of partial orders,
CL is the logic of singletons, which are partial orders of height 1.

$\mathrm{S4}$ is the logic of preorders,
$\mathrm{S5}$ is the logic of equivalence relations, which are preorders of height 1.

Let $\mathrm{L}[h]$ be the extension of a logic L with the axiom of height h.

$$\mathrm{IL}[1] \vdash \varphi \text{ iff } \mathrm{IL} \vdash \neg\neg\varphi \qquad \mathrm{S4}[1] \vdash \varphi \text{ iff } \mathrm{S4} \vdash \Diamond\Box\varphi$$

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$\mathrm{IL}[2] \vdash \varphi$ iff $\mathrm{IL} \vdash$?	$\mathrm{S4}[2] \vdash \varphi$ iff $\mathrm{S4} \vdash$?
$\mathrm{IL}[3] \vdash \varphi$ iff $\mathrm{IL} \vdash$?	$\mathrm{S4}[3] \vdash \varphi$ iff $\mathrm{S4} \vdash$?
...	...

Formulas and logics of finite height

intermediate:	$b^{\mathrm{I}}_0 = \bot,$	$b^{\mathrm{I}}_{i+1} = p_{i+1} \vee (p_{i+1} \to b^{\mathrm{I}}_i)$
modal:	$b_0 = \bot,$	$b_{i+1} = p_{i+1} \to \Box(\Diamond p_{i+1} \vee b_i)$

$\mathrm{L}[h]$ extends L with the formula of height h. In particular,

$$\mathrm{IL}[1] = \mathrm{CL}, \quad \mathrm{S4}[1] = \mathrm{S5}$$

The *k-canonical frame* of a logic L (the representation of the k-generated free algebra of L) is built from maximal L-consistent sets of k-formulas.

Shehtman, 1978: Let $k < \omega$. There exist formulas $\mathbf{B}_{h,k}$ (and their intuitionistic analogs $\mathbf{B}^{\mathrm{I}}_{h,k}$) such that for every x in the k-canonical frame F_k of $\mathrm{S4}$ (of Int)

$$\mathbf{B}_{h,k} \in x \quad \Longleftrightarrow \quad \text{the depth of } x \text{ in } F_k \text{ is less than or equal to } h.$$

Theorem

Let $k < \omega$. For all k-formulas φ we have:

- $\mathrm{IL}[h+1] \vdash \varphi$ iff $\mathrm{IL} \vdash \bigwedge_{i \le h}((\varphi \to \mathbf{B}^{\mathrm{I}}_{i,k}) \to \mathbf{B}^{\mathrm{I}}_{i,k});$
- $\mathrm{S4}[h+1] \vdash \varphi$ iff $\mathrm{S4} \vdash \bigwedge_{i \le h}(\Box(\Box\varphi \to \mathbf{B}_{i,k}) \to \mathbf{B}_{i,k}).$

In particular, for $h = 0$ the formulas $\mathbf{B}_{0,k}$ are $\bot$ for all $k < \omega$:

$$\mathrm{IL}[1] \vdash \varphi \text{ iff } \mathrm{IL} \vdash \neg\neg\varphi, \quad \mathrm{S4}[1] \vdash \varphi \text{ iff } \mathrm{S4} \vdash \Diamond\Box\varphi.$$

Analogs of the above translations exist whenever finite-height extensions of a logic are locally finite.

Segerberg, 1971; Kuznetsov, 1971; Komori, 1975: All $\mathrm{S4}[h]$, $\mathrm{IL}[h]$ are LF.

A logic is said to be *k-finite* if, up to the equivalence in it, there exist only finitely many k-formulas.
Hence, a logic is locally finite iff it is k-finite for every finite k.

L is *pretransitive* if there is a formula $\Diamond^*(p)$ ('master modality') s.t. $\Diamond^*(\varphi)$ expresses the satisfiability of φ in cones on models of L.

Pretransitive examples:

$\mathrm{K4}, \mathrm{GL}, \mathrm{wK4} = [\Diamond\Diamond p \to \Diamond p \vee p]$, $\mathrm{K5} = [\Diamond p \to \Box\Diamond p]$, $[\Diamond^n p \to \Diamond^m p]$ for $n > m$, (expanding) products of transitive logics

Shetman, Sh, 2016: Every 1-finite (a fortiori, locally finite) modal logic is a pretransitive logic of finite height.

Makinson, 1981: In general, the converse is not true!
There exists a pretransitive L s.t. none of the logics $\mathrm{L}[h]$, $h > 0$, are 1-finite: put $\mathrm{L} = [\Diamond^3 p \to \Diamond^2 p]$.

The *height of a polymodal frame* $(W, (R_i)_{i<n})$ is the height of the preorder $(W, (\bigcup_{i<n} R_i)^*)$.
In the pretransitive case, the formulas of finite height can be defined:
$B_0 = \bot,\ B_h = p_h \to \Box^*(\Diamond^* p_h \vee B_{h-1})$.

Theorem

Let L be a pretransitive logic, $h, k < \omega$. If $\mathrm{L}[h]$ is k-finite, then:

(a) *For every $i \leq h$, there exists a formula $\mathbf{B}_{i,k}$ such that $\mathbf{B}_{i,k} \in x$ iff the depth of x in the k-canonical frame of L is less than or equal to i.*

(b) *For all k-formulas φ,*

$$\mathrm{L}[h+1] \vdash \varphi \qquad \textit{iff} \qquad \mathrm{L} \vdash \bigwedge_{i \leq h} (\Box^*(\Box^* \varphi \to \mathbf{B}_{i,k}) \to \mathbf{B}_{i,k}).$$

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Theorem

Let L be a pretransitive logic, $h,k<\omega$. If $\mathrm{L}[h]$ is k-finite, then:

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Remark. The inconsistent logic $\mathrm{L}[0]$ is locally finite, hence for every pretransitive L

$$\mathrm{L}[1] \vdash \varphi \text{ iff } \mathrm{L} \vdash \Diamond^*\Box^*\varphi \tag{1}$$

Kudinov, Sh, 2011: A more direct (syntactic) proof of (1).

Contrary to the transitive case, k-finiteness of $\mathrm{L}[h]$ depends on h and k.

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This equivalence holds for many families of modal logics.

But it does not hold in general...

Theorem

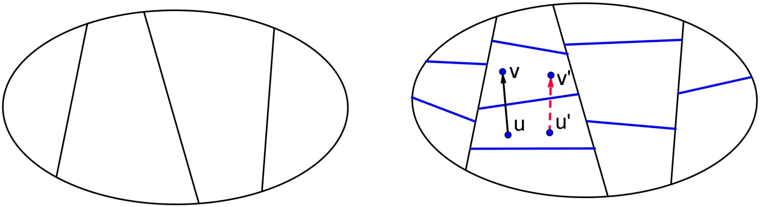
There exists a unimodal 1-finite logic which is not locally finite.

Proof.

Let $F = (\omega + 1, R)$, where xRy iff $x \leq y$ or $x = \omega$.
1-finiteness of the logic of F is a straightforward exercise.
Recall: If the logic of a frame is LF, then the logic of any its subframe is LF.
The restriction of the cluster $(\omega + 1, R)$ onto ω is the frame $(\omega, \leq)$, which is of infinite height. Thus $Log\,(\omega + 1, R)$ is not locally finite. $\square$

Corollary. There exists a unimodal 1-finite algebra which is not locally finite.

Proof. Consider free algebras of a non-locally finite, but 1-finite logic.

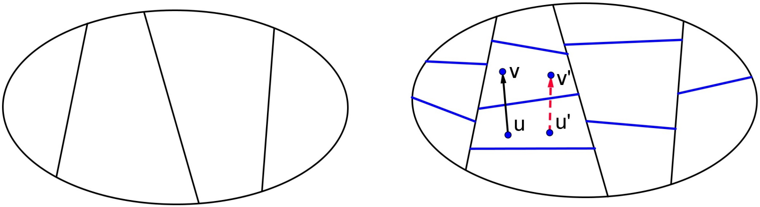


1-finiteness does not imply local finiteness.

Question

Does 2-finiteness of a modal logic imply local finiteness?
At least, does k-finiteness imply local finiteness, for some fixed k for all modal logics? For modal algebras?

The same questions are open in the intuitionistic case.



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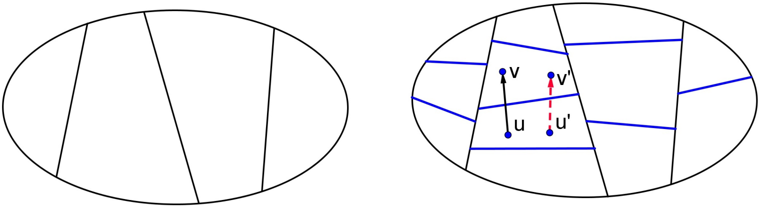

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Thank you!