Modal decision problems on sums of Kripke frames

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Sums of relational structures

Given a family (F_i : i in I) of relational structures (of the same signature) indexed by elements of another structure I, the *sum of* F_i*'s over* I is obtained from their disjoint union by connecting elements of i-th and j-th distinct components according to the relations in I.

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Not a new approach:

"Composition theorems" reduce the theory of a compound structure to theories (first-order, MSO) of components ([Mostowski, 1952], [Feferman–Vaught, 1959], [Shelah, 1975], [Gurevich, 1979], ...)

Fusions, or independent joins

For logics L_1, L_2 in disjoint modal vocabularies A, B,
$L_1 * L_2$ is the smallest logic in the vocabulary $\mathrm{A} \cup \mathrm{B}$, containing $L_1 \cup L_2$.

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[Kracht and Wolter, 1991; Fine and Schurz, 1996]:
If $L_1 = \mathrm{Log}\,\mathcal{F}$, $L_2 = \mathrm{Log}\,\mathcal{G}$, where $\mathcal{F}$ and $\mathcal{G}$ are classes of frames closed under disjoint unions, then $L_1 * L_2$ is the logic of the class
$\{(W, (R_a)_{a\in\mathrm{A}}, (S_b)_{b\in\mathrm{B}}) : (W, (R_a)_{a\in\mathrm{A}}) \in \mathcal{F}$ and $(W, (S_b)_{b\in\mathrm{B}}) \in \mathcal{G}\}$

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[Wolter, 1998]: If L_1 and L_2 are decidable, then $L_1 * L_2$ is decidable.

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Modal products

The *product* of two (unimodal, for simplicity) frames:
$(W, R) \times (V, S) = (W, R^\times, S^\times)$, where
$(w_1, v_1) R^\times (w_2, v_2)$ if $w_1 R w_2$ and $v_1 = v_2$,
$(w_1, v_1) S^\times (w_2, v_2)$ if $w_1 = w_2$ and $v_1 S v_2$.

$L_1 \times L_2$ is the logic of the class $\{\mathrm{F} \times \mathrm{G} : \mathrm{F} \models L_1$ and $\mathrm{G} \models L_2\}$.

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[Gabelaia, Kurucz, Wolter and Zakharyaschev, 2005]:
$\mathrm{S4} \times \mathrm{S4}$ is undecidable and lack the FMP.

In many cases, the sum operation preserves decidability/the finite model property/complexity of modal logics.

Three examples

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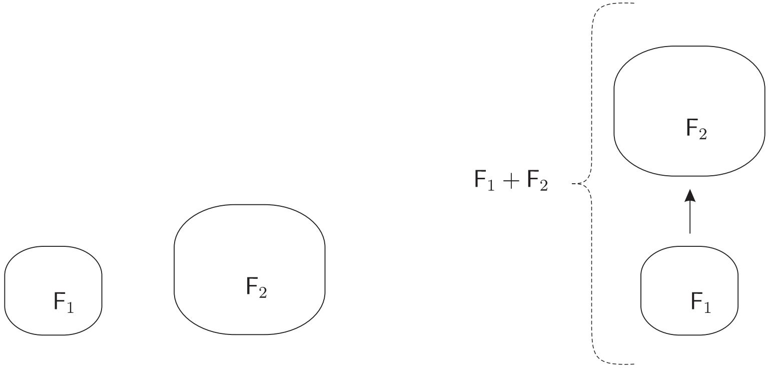
General phenomenon:

The modal satisfiability problem on sums over finite/Noetherian orders is $\leq_{\mathrm{T}}^{\mathrm{PSPACE}}$-reducible to the modal satisfiability problem on summands.

Sum of two frames

For $\mathsf{F}_1 = (W_1, R_1)$, $\mathsf{F}_2 = (W_2, R_2)$, $W_1 \cap W_2 = \varnothing$,

$$\mathsf{F}_1 + \mathsf{F}_2 = (W_1 \cup W_2, R_1 \cup R_2 \cup (W_1 \times W_2))$$



Sums of frames: unimodal case

Consider a family $(\mathsf{F}_i : i \in \mathsf{I})$ of frames indexed by elements of another frame I:
frame of indices $\mathsf{I} = (I, S)$;
frames-summands $\mathsf{F}_i = (W_i, R_i)$, i in I.

The *sum of the family* $(\mathsf{F}_i)_{i\in\mathsf{I}}$ *over* I is obtained from the disjoint union $\bigsqcup_{i\in I} \mathsf{F}_i$ by connecting elements of i-th and j-th distinct components according to I:

$$\sum_{i\in\mathsf{I}} \mathsf{F}_i = \left(\bigsqcup_{i\in I} W_i, R\right), \quad \text{where}$$

$$(i, w)R(j, v) \quad \text{iff} \quad i = j \,\&\, wR_iv \text{ or } i \neq j \,\&\, iSj.$$

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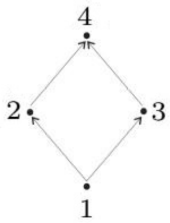
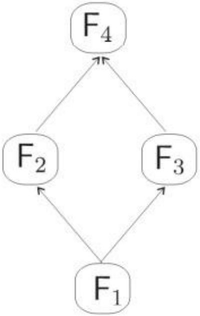


$\sum_{\mathsf{I}} \mathsf{F}_i$



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For classes $\mathcal{I}$, $\mathcal{F}$, $\sum_{\mathcal{I}} \mathcal{F}$ is the class of all sums $\sum_{i\in\mathsf{I}} \mathsf{F}_i$ such that $\mathsf{I} \in \mathcal{I}$ and $\mathsf{F}_i \in \mathcal{F}$ for every i in I.

Aim:

To transfer "good" properties of components to the logic $\mathrm{Log} \sum_{\mathcal{I}} \mathcal{F}$.

$A \leq_{\mathrm{T}}^{\mathrm{PSPACE}} B$ if there exists a polynomial space bounded oracle deterministic machine M with oracle B that recognizes A (it is assumed that every tape of M, including the oracle tape, is polynomial space bounded).

$\mathcal{F}^{[\forall]}$ is the class of frames in $\mathcal{F}$ enriched with the universal relation:
$\mathcal{F}^{[\forall]} = \{(W, W \times W, R) : (W, R) \in \mathcal{F}\}$

Theorem

Let $\mathcal{F}$ be a class of unimodal frames, $\mathcal{I}$ a class of Noetherian orders containing all finite trees. Then

$$\mathrm{SAT} \sum_{\mathcal{I}} \mathcal{F} \leq_{\mathrm{T}}^{\mathrm{PSPACE}} \mathrm{SAT}\, \mathcal{F}^{[\forall]}.$$

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$$\mathrm{SAT} \sum_{\mathcal{I}} \mathcal{F} \leq_{\mathrm{T}}^{\mathrm{PSPACE}} \mathrm{SAT}\, \mathcal{F}^{[\forall]}.$$

Remark on $[\forall]$. Let L be a transitive logic (or, more generally, a logic where transitive closure modality is expressible). If $\mathcal{F}$ a class of all (finite, or rooted, or finite rooted) frames of L, then $\mathrm{SAT}\, \mathcal{F}^{[\forall]} \leq_{\mathrm{T}}^{\mathrm{PSPACE}} \mathrm{SAT}\, \mathcal{F}$. (In fact, the reduction is stronger than $\leq_{\mathrm{T}}^{\mathrm{PSPACE}}$ [Spaan, 1996].) Hence, in these cases

$$\mathrm{SAT} \sum_{\mathcal{I}} \mathcal{F} \leq_{\mathrm{T}}^{\mathrm{PSPACE}} \mathrm{SAT}\, \mathcal{F}.$$

[Ladner, 1977] $\mathrm{S4}$, the modal logic of quasiorders, is in PSPACE.

Proof via sums: Every quasiorder F is isomorphic to the sum

$$\sum_{C \in \mathrm{sk}\mathsf{F}} (C, C \times C)$$

of its clusters over its skeleton $\mathrm{sk}\mathsf{F}$.

$\mathrm{S4}$ has the finite model property, hence $\mathrm{S4}$ is the logic of $\sum_{\text{finite PO}} \mathrm{Clusters}$.

Almost trivial: $\mathrm{SAT}(\mathrm{Clusters})$ is in NP, hence is in PSPACE.

Thus, we have:

$$\mathrm{SAT}(\mathrm{Quasiorders}) \leq_{\mathrm{T}}^{\mathrm{PSPACE}} \mathrm{SAT}(\mathrm{Clusters}) \in \mathrm{PSPACE}.$$

$\mathsf{F} = (W, R)$ is *weakly transitive* if $xRzRy \Rightarrow xRy \vee x = y$.
$\mathrm{wK4}$ is the logic of weakly transitive frames.

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Proof via sums: $\mathrm{wK4}$ has the FMP [Esakia, 1976 (2001); Shehtman, 2000].
Finite weakly transitive frames can be represented as $\sum_{\text{finite PO}} \mathcal{F}$, where (W, R) is in $\mathcal{F}$ iff R contains the difference relations (an easy exercise).
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In topological semantics ($\Diamond$ is for the derivation), $\mathrm{wK4}$ is the logic of all topological spaces.

[Bezhanishvili, Esakia, and Gabelaia, 2009] The logic of all T_0-spaces is the logic of finite weakly transitive frames where clusters contain at most one irreflexive point.

Corollary This logic is in PSPACE.

$\mathrm{GL}, \textsc{Grz}, \textsc{wGrz}$ are in PSPACE

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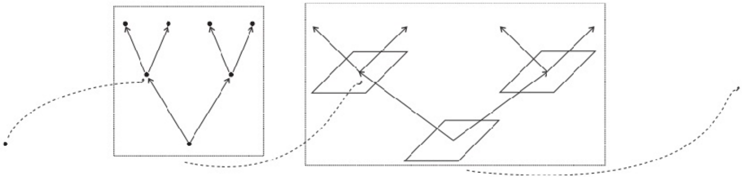
Proof is immediate from Kripke completeness:

Indices: Noetherian orders. Summands: $\{\bullet\}, \{\circ\}, \{\circ, \bullet\}$, respectively. $\square$

GLP is Kripke incomplete. However:

Beklemishev, 2007: There exists a polynomial-time translation f such that

$$\mathrm{GLP} \vdash \varphi \text{ iff } f(\varphi) \text{ is valid on the class of hereditary partial orderings:}$$



The logic J of hereditary partial orderings:

$\mathrm{J}{\restriction}0 =$ propositional logic, $\quad \mathrm{J}{\restriction}1 = \mathrm{GL} = \mathrm{Log}\,\mathrm{Noeth}$,

$\mathrm{J}{\restriction}2 = \mathrm{Log} \sum\limits_{\mathrm{Noeth}}^{\mathtt{lex}} \mathrm{Noeth}, \quad \mathrm{J}{\restriction}3 = \mathrm{Log} \sum\limits_{\mathrm{Noeth}}^{\mathtt{lex}} \left(\sum\limits_{\mathrm{Noeth}}^{\mathtt{lex}} \mathrm{Noeth} \right), \ldots$

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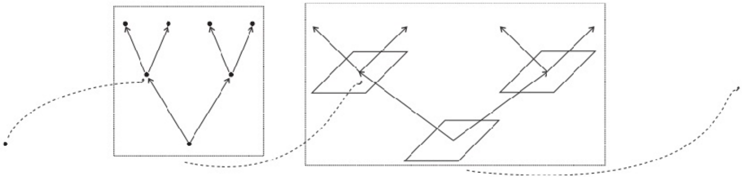

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Sh, 2008: J (and so GLP) is in PSPACE.

The same idea of the proof: Like in the case of GL, frames of J are obtained from a singleton via (iterated) sums over Noeth.

Fix $N \leq \omega$ and the N-modal signature $\{\Diamond_a : a < N\}$.

Frame of indices $\mathsf{I} = (I, (S_a)_{a<N})$;
frames-summands $\mathsf{F}_i = (W_i, (R_{i,a})_{a<N})$, i in I.

$$\sum_{i \in \mathsf{I}} \mathsf{F}_i = \left(\sqcup_{i \in I} W_i, (R_a)_{a<N}\right), \quad \text{where}$$

$$(i, w) R_a (j, v) \quad \text{iff} \quad i = j \,\&\, w R_{i,a} v \text{ or } i \neq j \,\&\, i S_a j.$$

Classes of N-frames $\mathcal{F}$ and $\mathcal{G}$ are said to be *interchangeable*, in symbols $\mathcal{F} \equiv \mathcal{G}$, if $\mathcal{F}$ and $\mathcal{G}$ have the same modal logic in the language enriched with the universal modality.

Formally, for an N-frame $\mathsf{F} = (W, R_0, R_1, \ldots)$, let $\mathsf{F}^{[\forall]}$ be the $(1+N)$-frame $(W, W \times W, R_0, R_1, \ldots)$. For a class $\mathcal{F}$ of N-frames, $\mathcal{F}^{[\forall]} = \{\mathsf{F}^{[\forall]} : \mathsf{F} \in \mathcal{F}\}$.

$\mathcal{F} \equiv \mathcal{G}$ if $\mathrm{Log}\,\mathcal{F}^{[\forall]} = \mathrm{Log}\,\mathcal{G}^{[\forall]}$.

Theorem (2018)

If $\mathcal{F} \equiv \mathcal{G}$, *then* $\sum_{\mathcal{I}} \mathcal{F} \equiv \sum_{\mathcal{I}} \mathcal{G}$.

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$\mathcal{F} \equiv \mathcal{G}$ if $\mathrm{Log}\,\mathcal{F}^{[\forall]} = \mathrm{Log}\,\mathcal{G}^{[\forall]}$.

Theorem (2018)

If $\mathcal{F} \equiv \mathcal{G}$, then $\sum_{\mathcal{I}} \mathcal{F} \equiv \sum_{\mathcal{I}} \mathcal{G}$.

Hence, if $\mathcal{F}$ and $\mathcal{G}$ are interchangeable, then the logics of sums $\sum_{\mathcal{I}} \mathcal{F}$ and $\sum_{\mathcal{I}} \mathcal{G}$ are equal; moreover, these classes of sums are interchangeable again, thus we have $\mathrm{Log} \sum_{\mathcal{J}} (\sum_{\mathcal{I}} \mathcal{F}) = \mathrm{Log} \sum_{\mathcal{J}} (\sum_{\mathcal{I}} \mathcal{G})$ for any other class of frames-indices $\mathcal{J}$, and so on:

Corollary

If $\mathcal{F} \equiv \mathcal{G}$, then for any $\mathcal{I}_1, \ldots, \mathcal{I}_s$ we have
$\mathrm{Log} \sum_{\mathcal{I}_1} \ldots \sum_{\mathcal{I}_s} \mathcal{F} = \mathrm{Log} \sum_{\mathcal{I}_1} \ldots \sum_{\mathcal{I}_s} \mathcal{G}$.

Corollary

If $\mathrm{Log}\,\mathcal{F}^{[\forall]}$ has the FMP, then for any classes $\mathcal{I}_1, \ldots, \mathcal{I}_s$ of finite frames the logic of the class $\sum_{\mathcal{I}_1} \ldots \sum_{\mathcal{I}_s} \mathcal{F}$ has the FMP.

Recall:

$\mathsf{I} = (I, (S_a)_{a<N})$; $\mathsf{F}_i = (W_i, (R_{i,a})_{a<N})$, i in I.

$$\sum_{i \in \mathsf{I}} \mathsf{F}_i = \left(\sqcup_{i \in I} W_i, (R_a)_{a<N}\right), \quad \text{where}$$

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a-sums

$(\mathsf{F}_i)_{i \in I}$ is a family of N-frames, $\mathsf{I} = (I, S)$ is a **unimodal** frame, $a < N$.
The *a-sum* ${}^{a}\sum_{\mathsf{I}} \mathsf{F}_i$ is the sum $\sum_{\mathsf{I}'} \mathsf{F}_i$, where I' is the N-frame whose domain is I, the a-th relation is S and other relations are empty.

Theorem (2008, 2018)

Let $\mathcal{F}$ be a class of N-frames, $a \in N$, $\mathcal{I}$ a class of Noetherian orders containing all finite trees. Then

$$\mathrm{Log}\ {}^{a}\!\!\sum_{\mathrm{Noeth}} \mathcal{F} = \mathrm{Log}\ {}^{a}\!\!\sum_{\mathrm{FinTr}} \mathcal{F} = \mathrm{Log}\ {}^{a}\!\!\sum_{\mathcal{I}} \mathcal{F}.$$

If also $\mathcal{I}$ is closed under finite disjoint unions, then

$${}^{a}\!\!\sum_{\mathrm{Noeth}} \mathcal{F} \equiv {}^{a}\!\!\sum_{\mathrm{FinTr}} \mathcal{F} \equiv {}^{a}\!\!\sum_{\mathcal{I}} \mathcal{F}.$$

Theorem (main result)

Let $L = \mathrm{Log}\ {}^{a_0}\sum_{\mathcal{I}_0} \ldots {}^{a_s}\sum_{\mathcal{I}_s} \mathcal{F}$, where
$\mathcal{F}$ is a class of N-frames, $a_0, \ldots, a_s < N < \omega$,
$\mathrm{FinTr} \subseteq \mathcal{I}_0, \ldots, \mathcal{I}_s \subseteq \mathrm{Noeth}$.
Then:

1. If $\mathrm{Log}\,\mathcal{F}^{[\forall]}$ has the finite model property, then so does L:

$$L = \mathrm{Log}\ {}^{a_0}\sum_{\mathrm{FinTr}} \ldots {}^{a_s}\sum_{\mathrm{FinTr}} \mathrm{FinFrames}\,\mathrm{Log}\,\mathcal{F}$$

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2. A formula φ is L-satisfiable iff φ is satisfiable in

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where $\mathrm{Tr}(\sharp\varphi)$ is the class of transitive trees with height and branching $\leq \sharp\varphi$.

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where $\mathrm{Tr}(\sharp\varphi)$ is the class of transitive trees with height and branching $\leq \sharp\varphi$.
3. $$\mathrm{SAT}\ {}^{a_0}\sum_{\mathcal{I}_0} \dots {}^{a_s}\sum_{\mathcal{I}_s} \mathcal{F} \ \leq_{\mathrm{T}}^{\mathrm{PSPACE}}\ \mathrm{SAT}\,\mathcal{F}^{[\forall]}.$$
If also classes $\mathcal{I}_a$ are closed under finite disjoint unions, then
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$N \leq \omega$.
$(\mathsf{F}_i)_{i\in I}$ is a family of N-frames, $\mathsf{I} = (I, S)$ is a **unimodal** frame.

The *lexicographic sum* $\sum\limits_{\mathsf{I}}^{\mathtt{lex}} \mathsf{F}_i$ is the $(1+N)$-frame $\left(\sqcup_{i\in I} W_i, S^{\mathtt{lex}}, (R_a)_{a<N}\right)$, where

$$\begin{aligned}(i,w)S^{\mathtt{lex}}(j,u) \quad & \text{iff} \quad iSj,\\ (i,w)R_a(j,u) \quad & \text{iff} \quad i=j \;\&\; wR_{i,a}u.\end{aligned}$$

Example. $\mathsf{S}_0 = (\{0\}, ())$ is a singleton frame in the 0-modal language.

For $a<\omega$, the a-modal fragment of J is characterized by $\underbrace{\sum\limits_{\mathrm{Noeth}}^{\mathtt{lex}} \ldots \sum\limits_{\mathrm{Noeth}}^{\mathtt{lex}}}_{a \text{ times}} \{\mathsf{S}_0\}$

For $\mathsf{F} = (W, (R_a)_{a<N})$, let $\mathsf{F}^{[\varnothing]}$ be the $(1+N)$ frame $(W, \varnothing, (R_a)_{a<N})$.

Simple fact. If I is irreflexive, then $\sum\limits_{\mathsf{I}}^{\mathtt{lex}} \mathsf{F}_i = \sum_{(I,S,\varnothing,\varnothing,\ldots)} \mathsf{F}_i^{[\varnothing]} = {}^{0}\sum_{\mathsf{I}} \mathsf{F}_i^{[\varnothing]}$.

If I is reflexive, then $\sum\limits_{\mathsf{I}}^{\mathtt{lex}} \mathsf{F}_i = \sum_{(I,S,\varnothing,\varnothing,\ldots)} \mathsf{F}_i^{[\forall]} = {}^{0}\sum_{\mathsf{I}} \mathsf{F}_i^{[\forall]}$.

Corollary. For all $a<\omega$, $\mathrm{SAT}\,(\underbrace{\sum\limits_{\mathrm{Noeth}}^{\mathtt{lex}} \ldots \sum\limits_{\mathrm{Noeth}}^{\mathtt{lex}}}_{a \text{ times}} \{\mathsf{S}_0\})^{[\forall]}$ is in PSPACE.

Concluding examples: lexicographic sums and products

The *lexicographic product* $\mathsf{I} \leftthreetimes \mathsf{F}$ is the is the sum $\sum\limits_{\mathsf{I}}^{\mathtt{lex}} \mathsf{F}_i$, where $\mathsf{F}_i = \mathsf{F}$ for all i in I.

For a class $\mathcal{I}$ of 1-frames and a class $\mathcal{G}$ of N-frames, the class $\mathcal{I} \leftthreetimes \mathcal{F}$ is the class of products $\mathsf{I} \leftthreetimes \mathsf{F}$ s.t. $\mathsf{I} \in \mathcal{I}$ and $\mathsf{F} \in \mathcal{F}$.
For logics L_1, L_2, $L_1 \leftthreetimes L_2 = \mathrm{Log}\,(\mathrm{Frames}\, L_1 \leftthreetimes \mathrm{Frames}\, L_2)$; likewise for sums.

$$\alpha = \Diamond_0 \Diamond_1 p \to \Diamond_0 p, \quad \beta = \Diamond_1 \Diamond_0 p \to \Diamond_0 p, \quad \gamma = \Diamond_0 p \to \Box_1 \Diamond_0 p.$$

[Balbiani, 2009] $\mathrm{S4} \leftthreetimes \mathrm{S4} = \mathrm{S4} * \mathrm{S4} + \{\alpha, \beta, \gamma\}$.

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[Sh, 2016] (1) If L is canonical, then $\sum\limits_{\mathrm{GL}}^{\mathtt{lex}} L = \mathrm{GL} * L + \{\alpha, \beta, \gamma\}$.

(2) If $L_1 * L_2 + \{\alpha, \beta, \gamma\}$ is Kripke complete, and L_1 is first-order definable without equality, then $\sum\limits_{L_1}^{\mathtt{lex}} L_2 = L_1 * L_2 + \{\alpha, \beta, \gamma\}$.

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FinQO denotes finite quasiorders. $\mathcal{C}$ denotes finite frames of form $(W, \varnothing, W \times W)$.

[Sh, 2018] $\mathrm{S4} \leftthreetimes \mathrm{S4}$ and $\mathrm{GL} \leftthreetimes \mathrm{S4}$ have the fmp, moreover — product/sum fmp:

$$\mathrm{S4} \leftthreetimes \mathrm{S4} = \sum_{\mathrm{S4}}^{\texttt{lex}} \mathrm{S4} = \mathrm{Log}\ {}^{0}\!\sum_{\mathrm{FinTr}} \mathrm{FinQO}^{[\forall]}$$

$$\mathrm{GL} \leftthreetimes \mathrm{S4} = \sum_{\mathrm{GL}}^{\texttt{lex}} \mathrm{S4} = \mathrm{Log}\ {}^{0}\!\sum_{\mathrm{FinTr}} {}^{1}\!\sum_{\mathrm{FinTr}} \mathcal{C}.$$

Corollary $\mathrm{S4} \leftthreetimes \mathrm{S4}$ and $\mathrm{GL} \leftthreetimes \mathrm{S4}$ are PSPACE-complete.

Further results/directions:

- Further FMP results

  For lexicographic sums and products, filtrations can be reconstructed from filtrations on indices/summands.

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  This positive result seems to be scalable. Filtrations give FMP, and also we have the following fact: φ is satisfiable in sums over finite linear (quasi)orders iff φ is satisfiable in such sums with the size of orders $\leq \sharp\varphi$.

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- Sum-based operations in the Kripke-incomplete case

  Logics of sums are Kripke complete. What could be the definition of sums for modal algebras (general Kripke frames)?

  E..g., can we approximate GLP by sums like $\sum_{\mathrm{Noeth}} \dots \sum_{\mathrm{Noeth}} \mathcal{F}_0$?

Sum-based operations seem to provide a nice way of combining modal logics.

Thank you!