Applications of filtrations: PDLization and local finiteness

Ilya Shapirovsky

New Mexico State University

Duality, Order, (Co)algebras, Topology, and Related topics
July 9, 2021

This talk is about the finite model property of propositional normal modal logics.

normal modal logics $\supsetneq$

Kripke complete logics $\supsetneq$

logics with the finite model property

This talk is about the finite model property of propositional normal modal logics.

normal modal logics $\supsetneq$

Kripke complete logics $\supsetneq$

logics with the finite model property $\supsetneq$

logics that admit filtration $\supsetneq$

locally finite logics

Part I. If a logic admits filtration then we can enrich it with the converse and transitive closure modalities and modalities for union and composition, preserving the finite model property.

1. S. Kikot, E. Zolin, Sh
   Filtration safe operations on frames. In *Advances in Modal Logic*, volume 10, pages 333–352, 2014.
2. S. Kikot, E. Zolin, Sh
   Modal logics with transitive closure: completeness, decidability, filtration. In *Advances in Modal Logic*, volume 13, pages 369–388, 2020.

Part II. Local finiteness of modal logics/algebras via filtrations.

1. V. Shehtman, Sh
   Local tabularity without transitivity. In *Advances in Modal Logic*, volume 11, pages 520–534, 2016.
2. Sh
   Modal logics of finite direct powers of ω have the finite model property. In WoLLIC 2019, Lecture Notes in Computer Science, pages 610–618, 2019.

Unimodal language: a countable set $\textsc{Var}$ (propositional variables), Boolean connectives, a unary connective $\Diamond$ ($\Box$ abbreviates $\neg\Diamond\neg$).

Normal modal logics: Definition 1

A set of modal formulas L is a *normal modal logic* if L contains

- all tautologies
- $\Diamond\bot \leftrightarrow \bot$, $\quad \Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$

and is closed under MP, Sub, and Mon:
if $(\varphi \to \psi) \in \mathrm{L}$, then $(\Diamond\varphi \to \Diamond\psi) \in \mathrm{L}$.

Normal modal logics: Definition 2

A modal algebra is a BA endowed with a unary operation that distributes over finite disjunctions.

A set of modal formulas L is a *normal modal logic* if L is the logic of a modal algebra A: $\mathrm{L} = \{\varphi \mid A \vDash \varphi = \top\}$

Unimodal language: a countable set $\textsc{Var}$ (propositional variables), Boolean connectives, a unary connective $\Diamond$ ($\Box$ abbreviates $\neg\Diamond\neg$).

Normal modal logics: Definition 1

A set of modal formulas L is a *normal modal logic* if L contains

- all tautologies
- $\Diamond\bot \leftrightarrow \bot, \quad \Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$

and is closed under MP, Sub, and Mon:
if $(\varphi \to \psi) \in \mathrm{L}$, then $(\Diamond\varphi \to \Diamond\psi) \in \mathrm{L}$.

Normal modal logics: Definition 2

A *modal algebra* is a BA endowed with a unary operation that distributes over finite disjunctions.

A set of modal formulas L is a *normal modal logic* if L is the logic of a modal algebra A: $\mathrm{L} = \{\varphi \mid A \vDash \varphi = \top\}$

Kripke semantics

A *(Kripke) frame* F is a pair (W, R), where $W \neq \varnothing$, $R \subseteq W \times W$.
A *model* M on F is a pair (F, θ) where $\theta : \textsc{Var} \to \mathcal{P}(W)$.

$$M, x \vDash p \text{ iff } x \in \theta(p), \qquad M, x \vDash \Diamond\varphi \text{ iff } M, y \vDash \varphi \text{ for some } y \text{ with } xRy.$$

Log$(F)$$= \{\varphi \mid F \vDash \varphi\}$, where $F \vDash \varphi$ means that $M, x \vDash \varphi$ for every M on F and every x in M.

The *algebra* $Alg(F)$ *of a frame* $F = (W, R)$ is the modal algebra $(\mathcal{P}(W), R^{-1})$.

$$\text{Hence: } F \vDash \varphi \text{ iff } Alg(F) \vDash \varphi = \top.$$

A logic L is *Kripke complete* if L is the logic of a class $\mathcal{C}$ of Kripke frames: $\mathrm{L} = \bigcap\{Log(F) \mid F \in \mathcal{C}\}$.

A logic L has the *finite model property* if L is the logic of a class $\mathcal{C}$ of finite models (algebras, frames).

If a logic L has the fmp and the class of its finite frames (algebras) is decidable, then L is co-RE.

In particular, if L has the fmp and is finitely axiomatizable, then it is decidable.

Example

[McKinsey, 1941] The logic $\mathrm{S4} = [p \to \Diamond p, \Diamond\Diamond p \to \Diamond p]$ has the fmp and hence is decidable.

A general problem:

To increase expressive power of a logic L preserving positive results on decidability/completeness.

A general problem:

To increase expressive power of a logic L preserving positive results on decidability/completeness.

Assume that we want to enrich a modal logic L with:

- *universal modality* — corresponds to $S = W \times W$, or

A general problem:

To increase expressive power of a logic L preserving positive results on decidability/completeness.

Assume that we want to enrich a modal logic L with:

- *universal modality* — corresponds to $S = W \times W$, or
- *converse modality* — corresponds to $S = R^{-1}$, or

A general problem:

To increase expressive power of a logic L preserving positive results on decidability/completeness.

Assume that we want to enrich a modal logic L with:

- *universal modality* — corresponds to $S = W \times W$, or
- *converse modality* — corresponds to $S = R^{-1}$, or
- *transitive closure modality* — corresponds to $S = R^{+}$, or

A general problem:

To increase expressive power of a logic L preserving positive results on decidability/completeness.

Assume that we want to enrich a modal logic L with:

- *universal modality* — corresponds to $S = W \times W$, or
- *converse modality* — corresponds to $S = R^{-1}$, or
- *transitive closure modality* — corresponds to $S = R^{+}$, or
- modalities for *union*, *composition* of relations, etc.

A general problem:

To increase expressive power of a logic L preserving positive results on decidability/completeness.

Assume that we want to enrich a modal logic L with:

- *universal modality* — corresponds to $S = W \times W$, or
- *converse modality* — corresponds to $S = R^{-1}$, or
- *transitive closure modality* — corresponds to $S = R^{+}$, or
- modalities for *union*, *composition* of relations, etc.

This means:
– add a new modality [*new*] to the language of L, and
– add new axioms to L that govern the behaviour of [*new*].

A general problem:

To increase expressive power of a logic L preserving positive results on decidability/completeness.

Assume that we want to enrich a modal logic L with:

- *universal modality* — corresponds to $S = W \times W$, or
- *converse modality* — corresponds to $S = R^{-1}$, or
- *transitive closure modality* — corresponds to $S = R^{+}$, or
- modalities for *union*, *composition* of relations, etc.

This means:
– add a new modality [*new*] to the language of L, and
– add new axioms to L that govern the behaviour of [*new*].

Question

Which properties of the logic L are preserved?

The transitive closure of a binary relation R is denoted by R^+.
Given a frame $F = (W, R)$, we write $F^{(+)} = (W, R, R^+)$.
For a class $\mathcal{F}$ of frames, denote $\mathcal{F}^{(+)} = \{F^{(+)} \mid F \in \mathcal{F}\}$.

The extension of a normal unimodal logic L with the *transitive closure modality* is the minimal normal bimodal logic L^+ that contains L and the axioms [Segerberg, 1970s]:

$$\text{(A1)} \quad \boxplus p \to \Box p \qquad \text{(A2)} \quad \boxplus p \to \Box \boxplus p \qquad \text{(A3)} \quad \Box p \wedge \boxplus(p \to \Box p) \to \boxplus p.$$

Proposition

$(W, R, S) \models \text{(A1)} \wedge \text{(A2)} \wedge \text{(A3)}$ *iff* $S = R^+$.

Proposition

$$\mathrm{Frames}(\mathrm{L}^+) = \mathrm{Frames}(\mathrm{L})^{(+)}.$$

Are decidability, the FMP, Kripke completeness of a logic preserved?

In general, no.

Counterexamples:

- L is decidable $\quad \not\Longrightarrow \quad \mathrm{L}^{\mathsf{u}}$ is decidable [Spaan, 1996]
  (here L^{u} extends L with the *universal modality*)

Are decidability, the FMP, Kripke completeness of a logic preserved?

In general, no.

Counterexamples:

- L is decidable $\quad\not\Longrightarrow\quad$ $\mathrm{L}^{\mathbf{u}}$ is decidable [Spaan, 1996]
  (here $\mathrm{L}^{\mathbf{u}}$ extends L with the *universal modality*)
- L has the FMP $\quad\not\Longrightarrow\quad$ $\mathrm{L}^{\mathbf{t}}$ has the FMP [Wolter, 1995]
  (here $\mathrm{L}^{\mathbf{t}}$ extends L with the *converse modality*)

Are decidability, the FMP, Kripke completeness of a logic preserved?

In general, no.

Counterexamples:

- L is decidable $\quad\not\Longrightarrow\quad$ $\mathrm{L}^{\mathbf{u}}$ is decidable [Spaan, 1996]
  (here $\mathrm{L}^{\mathbf{u}}$ extends L with the *universal modality*)
- L has the FMP $\quad\not\Longrightarrow\quad$ $\mathrm{L}^{\mathbf{t}}$ has the FMP [Wolter, 1995]
  (here $\mathrm{L}^{\mathbf{t}}$ extends L with the *converse modality*)
- L is decidable $\quad\not\Longrightarrow\quad$ L^{+} is decidable [Spaan, 1996]
  (here L^{+} extends L with the *transitive closure modality*)

Are decidability, the FMP, Kripke completeness of a logic preserved?

In general, no.

Counterexamples:

- L is decidable $\quad\not\Longrightarrow\quad$ $\mathrm{L}^{\mathbf{u}}$ is decidable [Spaan, 1996]
  (here $\mathrm{L}^{\mathbf{u}}$ extends L with the *universal modality*)
- L has the FMP $\quad\not\Longrightarrow\quad$ $\mathrm{L}^{\mathbf{t}}$ has the FMP [Wolter, 1995]
  (here $\mathrm{L}^{\mathbf{t}}$ extends L with the *converse modality*)
- L is decidable $\quad\not\Longrightarrow\quad$ L^{+} is decidable [Spaan, 1996]
  (here L^{+} extends L with the *transitive closure modality*)
- L has the FMP $\quad\not\Longrightarrow\quad$ $\mathrm{L}^{\mathbf{u}}$ has the FMP [Wolter, 1994]

Are decidability, the FMP, Kripke completeness of a logic preserved?

In general, no.

Counterexamples:

- L is decidable $\quad\not\Longrightarrow\quad$ $\mathrm{L}^{\mathbf{u}}$ is decidable [Spaan, 1996]
  (here $\mathrm{L}^{\mathbf{u}}$ extends L with the *universal modality*)
- L has the FMP $\quad\not\Longrightarrow\quad$ $\mathrm{L}^{\mathbf{t}}$ has the FMP [Wolter, 1995]
  (here $\mathrm{L}^{\mathbf{t}}$ extends L with the *converse modality*)
- L is decidable $\quad\not\Longrightarrow\quad$ L^{+} is decidable [Spaan, 1996]
  (here L^{+} extends L with the *transitive closure modality*)
- L has the FMP $\quad\not\Longrightarrow\quad$ $\mathrm{L}^{\mathbf{u}}$ has the FMP [Wolter, 1994]
- L is Kripke complete $\quad\not\Longrightarrow\quad$ $\mathrm{L}^{\mathbf{u}}$ is Kripke complete [Kracht, 1999]
- .....

Are decidability, the FMP, Kripke completeness of a logic preserved?

In general, no.

Counterexamples:

- L is decidable $\quad\not\Longrightarrow\quad$ $\mathrm{L}^{\mathbf{u}}$ is decidable [Spaan, 1996]
  (here $\mathrm{L}^{\mathbf{u}}$ extends L with the *universal modality*)
- L has the FMP $\quad\not\Longrightarrow\quad$ $\mathrm{L}^{\mathbf{t}}$ has the FMP [Wolter, 1995]
  (here $\mathrm{L}^{\mathbf{t}}$ extends L with the *converse modality*)
- L is decidable $\quad\not\Longrightarrow\quad$ L^{+} is decidable [Spaan, 1996]
  (here L^{+} extends L with the *transitive closure modality*)
- L has the FMP $\quad\not\Longrightarrow\quad$ $\mathrm{L}^{\mathbf{u}}$ has the FMP [Wolter, 1994]
- L is Kripke complete $\quad\not\Longrightarrow\quad$ $\mathrm{L}^{\mathbf{u}}$ is Kripke complete [Kracht, 1999]
- .....

Filtrability is preserved!

For a model M and a set of formulas Γ,
$x \sim_\Gamma y \quad \leftrightharpoons \quad \forall\psi \in \Gamma\ (M, x \models \psi \Leftrightarrow M, y \models \psi)$.

Definition (Filtration)

Let Γ be a subformula-closed set of formulas. A *filtration* of a model $M = (W, R, \theta)$ through Γ (or *Γ-filtration*, for short) is a model $\widehat{M} = (\widehat{W}, \widehat{R}, \widehat{\theta})$ s.t.

1. $\widehat{W} = W/\sim$ for some equivalence relation $\sim$ such that $\sim\ \subseteq\ \sim_\Gamma$, i.e.,

   $$\text{if } x \sim y, \text{ then } \forall\psi \in \Gamma\ (M, x \models \psi \Leftrightarrow M, y \models \psi).$$

2. $\widehat{M}, \widehat{x} \models p \Leftrightarrow M, x \models p$ for all $p \in \Gamma$
   Here $\widehat{x}$ is the class of x modulo $\sim$.

3. $R_\sim \ \subseteq\ \widehat{R}\ \subseteq\ R_\sim^\Gamma$, where

   $$\widehat{x}\, R_\sim\, \widehat{y} \quad \leftrightharpoons \quad \exists x' \sim x\ \exists y' \sim y\ (x'\, R\, y')$$
   $$\widehat{x}\, R_\sim^\Gamma\, \widehat{y} \quad \leftrightharpoons \quad \forall\psi\ (\Diamond\psi \in \Gamma\ \&\ M, y \models \psi \ \Rightarrow\ M, x \models \Diamond\psi)$$

   The relations $R_\sim$ and $R_\sim^\Gamma$ on $\widehat{W}$ are called the *minimal* and the *maximal filtered relations*, respectively.

Filtration lemma

$$\forall\psi \in \Gamma \quad (M, x \models \psi \ \Leftrightarrow\ \widehat{M}, \widehat{x} \models \psi)$$

For a model M and a set of formulas Γ,
$x \sim_\Gamma y \quad \leftrightharpoons \quad \forall \psi \in \Gamma\ (M, x \models \psi \Leftrightarrow M, y \models \psi)$.

Definition (Filtration)

Let Γ be a subformula-closed set of formulas. A *filtration* of a model $M = (W, R, \theta)$ through Γ (or *Γ-filtration*, for short) is a model $\widehat{M} = (\widehat{W}, \widehat{R}, \widehat{\theta})$ s.t.

1. $\widehat{W} = W/\sim$ for some equivalence relation $\sim$ such that $\sim \subseteq \sim_\Gamma$, i.e.,

   $$\text{if } x \sim y, \text{ then } \forall \psi \in \Gamma\ (M, x \models \psi \Leftrightarrow M, y \models \psi).$$

2. $\widehat{M}, \widehat{x} \models p \Leftrightarrow M, x \models p$ for all $p \in \Gamma$
   Here $\widehat{x}$ is the class of x modulo $\sim$.

3. $R_\sim \ \subseteq\ \widehat{R}\ \subseteq\ R^\Gamma_\sim$, where

   $$\widehat{x}\, R_\sim\, \widehat{y} \quad \leftrightharpoons \quad \exists x' \sim x\ \exists y' \sim y\ (x'\, R\, y')$$
   $$\widehat{x}\, R^\Gamma_\sim\, \widehat{y} \quad \leftrightharpoons \quad \forall \psi\ (\Diamond\psi \in \Gamma\ \&\ M, y \models \psi \ \Rightarrow\ M, x \models \Diamond\psi)$$

   The relations $R_\sim$ and $R^\Gamma_\sim$ on $\widehat{W}$ are called the *minimal* and the *maximal filtered relations*, respectively.

Filtration lemma

$$\forall \psi \in \Gamma \quad (M, x \models \psi \ \Leftrightarrow\ \widehat{M}, \widehat{x} \models \psi)$$

For a model M and a set of formulas Γ,
$x \sim_\Gamma y \quad :\Leftrightarrow \quad \forall \psi \in \Gamma\ (M, x \models \psi \Leftrightarrow M, y \models \psi)$.

Definition (Filtration)

Let Γ be a subformula-closed set of formulas. A *filtration* of a model $M = (W, R, \theta)$ through Γ (or *Γ-filtration*, for short) is a model $\widehat{M} = (\widehat{W}, \widehat{R}, \widehat{\theta})$ s.t.

1. $\widehat{W} = W/{\sim}$ for some equivalence relation $\sim$ such that ${\sim} \subseteq {\sim_\Gamma}$, i.e.,

   if $x \sim y$, then $\forall \psi \in \Gamma\ (M, x \models \psi \Leftrightarrow M, y \models \psi)$.

2. $\widehat{M}, \widehat{x} \models p \Leftrightarrow M, x \models p$ for all $p \in \Gamma$

   Here $\widehat{x}$ is the class of x modulo $\sim$.

3. $R_\sim \ \subseteq\ \widehat{R}\ \subseteq\ R^\Gamma_\sim$, where

   $$\widehat{x}\, R_\sim\, \widehat{y} \quad :\Leftrightarrow \quad \exists x' \sim x\ \exists y' \sim y\ (x'\, R\, y')$$
   $$\widehat{x}\, R^\Gamma_\sim\, \widehat{y} \quad :\Leftrightarrow \quad \forall \psi\ (\Diamond\psi \in \Gamma\ \&\ M, y \models \psi\ \Rightarrow\ M, x \models \Diamond\psi)$$

   The relations $R_\sim$ and $R^\Gamma_\sim$ on $\widehat{W}$ are called the *minimal* and the *maximal filtered relations*, respectively.

Filtration lemma

$$\forall \psi \in \Gamma \quad (M, x \models \psi \ \Leftrightarrow\ \widehat{M}, \widehat{x} \models \psi)$$

To get a finite L-model, we have two parameters to choose: $\sim$, $\widehat{R}$

FMP via filtrations

Construct finite filtrations

- of the canonical model,
- or of any other models characterizing the logic,
- in particular, of models based on frames of the logic, provided that the logic is Kripke complete.

For a model M and a set of formulas Γ,
$x \sim_\Gamma y \quad \leftrightharpoons \quad \forall \psi \in \Gamma \, (M, x \models \psi \Leftrightarrow M, y \models \psi)$.

Definition (Filtration)

Let Γ be a subformula-closed set of formulas. A *filtration* of a model $M = (W, R, \theta)$ through Γ (or *Γ-filtration*, for short) is a model $\widehat{M} = (\widehat{W}, \widehat{R}, \widehat{\theta})$ s.t.

1. $\widehat{W} = W/\sim$ for some equivalence relation $\sim$ such that $\sim \subseteq \sim_\Gamma$, i.e.,

   $$\text{if } x \sim y, \text{ then } \forall \psi \in \Gamma \, (M, x \models \psi \Leftrightarrow M, y \models \psi).$$

2. $\widehat{M}, \widehat{x} \models p \Leftrightarrow M, x \models p$ for all $p \in \Gamma$

   Here $\widehat{x}$ is the class of x modulo $\sim$.

3. $R_\sim \;\subseteq\; \widehat{R} \;\subseteq\; R^\Gamma_\sim$, where

   $$\widehat{x} \, R_\sim \, \widehat{y} \quad \leftrightharpoons \quad \exists x' \sim x \; \exists y' \sim y \; (x' \, R \, y')$$
   $$\widehat{x} \, R^\Gamma_\sim \, \widehat{y} \quad \leftrightharpoons \quad \forall \psi \, (\Diamond \psi \in \Gamma \,\&\, M, y \models \psi \;\Rightarrow\; M, x \models \Diamond \psi)$$

   The relations $R_\sim$ and $R^\Gamma_\sim$ on $\widehat{W}$ are called the *minimal* and the *maximal filtered relations*, respectively.

Filtration lemma

$$\forall \psi \in \Gamma \quad (M, x \models \psi \;\Leftrightarrow\; \widehat{M}, \widehat{x} \models \psi)$$

To get a finite L-model, we have two parameters to choose: $\sim$, $\widehat{R}$

FMP via filtrations

Construct finite filtrations

- of the canonical model,
- or of any other models characterizing the logic,
- in particular, of models based on frames of the logic, provided that the logic is Kripke complete.

Examples [Lemmon, Scott, Segerberg, Gabbay, Shehtman,...]

For a model M and a set of formulas Γ,
$x \sim_\Gamma y \quad \leftrightharpoons \quad \forall\psi \in \Gamma\ (M, x \models \psi \Leftrightarrow M, y \models \psi)$.

Definition (Filtration)

Let Γ be a subformula-closed set of formulas. A *filtration* of a model $M = (W, R, \theta)$ through Γ (or *Γ-filtration*, for short) is a model $\widehat{M} = (\widehat{W}, \widehat{R}, \widehat{\theta})$ s.t.

1. $\widehat{W} = W/\sim$ for some equivalence relation $\sim$ such that $\sim\ \subseteq\ \sim_\Gamma$, i.e.,

   if $x \sim y$, then $\forall\psi \in \Gamma\ (M, x \models \psi \Leftrightarrow M, y \models \psi)$.

2. $\widehat{M}, \widehat{x} \models p \Leftrightarrow M, x \models p$ for all $p \in \Gamma$

   Here $\widehat{x}$ is the class of x modulo $\sim$.

3. $R_\sim \ \subseteq\ \widehat{R}\ \subseteq\ R^\Gamma_\sim$, where

$$\widehat{x}\, R_\sim\, \widehat{y} \quad \leftrightharpoons \quad \exists x' \sim x\ \exists y' \sim y\ (x'\, R\, y')$$
$$\widehat{x}\, R^\Gamma_\sim\, \widehat{y} \quad \leftrightharpoons \quad \forall\psi\ (\Diamond\psi \in \Gamma\ \&\ M, y \models \psi \ \Rightarrow\ M, x \models \Diamond\psi)$$

   The relations $R_\sim$ and $R^\Gamma_\sim$ on $\widehat{W}$ are called the *minimal* and the *maximal filtered relations*, respectively.

Filtration lemma

$$\forall\psi \in \Gamma \quad (M, x \models \psi \ \Leftrightarrow\ \widehat{M}, \widehat{x} \models \psi)$$

To get a finite L-model, we have two parameters to choose: $\sim$, $\widehat{R}$

FMP via filtrations

Construct finite filtrations

- of the canonical model,
- or of any other models characterizing the logic,
- in particular, of models based on frames of the logic, provided that the logic is Kripke complete.

Examples [Lemmon, Scott, Segerberg, Gabbay, Shehtman,...]

- K, $\mathrm{T} = [p \to \Diamond p]$, $\mathrm{KB} = [p \to \Box\Diamond p]$, $[\Diamond p \to \Diamond \ldots \Diamond p]$

  Very simple: use Kripke completeness and put $\sim\, = \,\sim_\Gamma$, $\widehat{R} = R_\sim$.

- $\mathrm{K4} = [\Diamond\Diamond p \to \Diamond p]$; $\mathrm{K4.2} = [\Diamond\Diamond p \to \Diamond p,\ \Diamond\Box p \to \Box\Diamond p]$

  Simple: consider $\sim_\Gamma$ and the transitive closure of $R_\sim$; for $\mathrm{K4.2}$, assume that M is rooted.

For a model M and a set of formulas Γ,
$x \sim_\Gamma y \quad \leftrightharpoons \quad \forall\psi \in \Gamma\ (M, x \models \psi \Leftrightarrow M, y \models \psi)$.

Definition (Filtration)

Let Γ be a subformula-closed set of formulas. A *filtration* of a model $M = (W, R, \theta)$ through Γ (or *Γ-filtration*, for short) is a model $\widehat{M} = (\widehat{W}, \widehat{R}, \widehat{\theta})$ s.t.

1. $\widehat{W} = W/\!\sim$ for some equivalence relation $\sim$ such that $\sim\ \subseteq\ \sim_\Gamma$, i.e.,

   if $x \sim y$, then $\forall\psi \in \Gamma\ (M, x \models \psi \Leftrightarrow M, y \models \psi)$.

2. $\widehat{M}, \widehat{x} \models p \Leftrightarrow M, x \models p$ for all $p \in \Gamma$

   Here $\widehat{x}$ is the class of x modulo $\sim$.

3. $R_\sim \ \subseteq\ \widehat{R}\ \subseteq\ R^\Gamma_\sim$, where

   $$\widehat{x}\, R_\sim\, \widehat{y} \quad \leftrightharpoons \quad \exists x' \sim x\ \exists y' \sim y\ (x'\, R\, y')$$
   $$\widehat{x}\, R^\Gamma_\sim\, \widehat{y} \quad \leftrightharpoons \quad \forall\psi\ (\Diamond\psi \in \Gamma\ \&\ M, y \models \psi\ \Rightarrow\ M, x \models \Diamond\psi)$$

   The relations $R_\sim$ and $R^\Gamma_\sim$ on $\widehat{W}$ are called the *minimal* and the *maximal filtered relations*, respectively.

Filtration lemma

$$\forall\psi \in \Gamma \quad (M, x \models \psi \ \Leftrightarrow\ \widehat{M}, \widehat{x} \models \psi)$$

To get a finite L-model, we have two parameters to choose: $\sim$, $\widehat{R}$

FMP via filtrations

Construct finite filtrations

- of the canonical model,
- or of any other models characterizing the logic,
- in particular, of models based on frames of the logic, provided that the logic is Kripke complete.

Examples [Lemmon, Scott, Segerberg, Gabbay, Shehtman,...]

- K, $\mathrm{T} = [p \to \Diamond p]$, $\mathrm{KB} = [p \to \Box\Diamond p]$, $[\Diamond p \to \Diamond \ldots \Diamond p]$

  Very simple: use Kripke completeness and put $\sim\ =\ \sim_\Gamma$, $\widehat{R} = R_\sim$.
- $\mathrm{K4} = [\Diamond\Diamond p \to \Diamond p]$; $\mathrm{K4.2} = [\Diamond\Diamond p \to \Diamond p,\ \Diamond\Box p \to \Box\Diamond p]$

  Simple: consider $\sim_\Gamma$ and the transitive closure of $R_\sim$; for $\mathrm{K4.2}$, assume that M is rooted.
- $\mathrm{S4.1} = \mathrm{S4} + \Box\Diamond p \to \Diamond\Box p$; $[\Diamond \ldots \Diamond p \to \Diamond p]$

  Require more steps. In particular, $\sim_\Gamma$ should be refined.
- Some products, expanding products...

  Constructions might be very complicated.

A class of frames $\mathcal{F}$ *admits filtration* if, for any finite Sub-closed set of formulas Γ and an $\mathcal{F}$-model M, there exists a finite $\mathcal{F}$-model that is a Γ-filtration of M.
A class of models $\mathcal{M}$ *admits filtration* if, for any finite Sub-closed set of formulas Γ and any $M \in \mathcal{M}$, there is a finite model in $\mathcal{M}$ that is a Γ-filtration of M.

Various versions of the AF property:
[Goranko & Passy 1992]
[G. Bezhanishvili & Zakharyaschev, 1997]
[N. Bezhanishvili & Ten Cate, 2006]
[Schmidt & Tishkovsky, 2008]
[Ilin, 2018]
...

If L is Kripke complete and the class of its frames Frames(L) admits filtration, then L has the FMP.

If the class of models Mod(L) of a logic L admits filtration, then L has the FMP and hence is Kripke complete.

A class of frames $\mathcal{F}$ *admits filtration* if, for any finite Sub-closed set of formulas Γ and an $\mathcal{F}$-model M, there exists a finite $\mathcal{F}$-model that is a Γ-filtration of M.
A class of models $\mathcal{M}$ *admits filtration* if, for any finite Sub-closed set of formulas Γ and any $M \in \mathcal{M}$, there is a finite model in $\mathcal{M}$ that is a Γ-filtration of M.

Various versions of the AF property:
[Goranko & Passy 1992]
[G. Bezhanishvili & Zakharyaschev, 1997]
[N. Bezhanishvili & Ten Cate, 2006]
[Schmidt & Tishkovsky, 2008]
[Ilin, 2018]
...

If L is Kripke complete and the class of its frames Frames(L) admits filtration, then L has the FMP.

If the class of models Mod(L) of a logic L admits filtration, then L has the FMP and hence is Kripke complete.

Theorem (Filtration safe operations on frames)

Suppose that a class $\mathcal{F}$ of frames admits filtration. Then the classes $\mathcal{F}^u$, $\mathcal{F}^t$, $\mathcal{F}^+$ admit filtrations too.

Corollary

Suppose that the class of frames of a logic L admits filtration. Then L^u, L^t, L^+ have the fmp provided that they are Kripke complete.

The statements about the universal modality are due to [Goranko & Passy, 1992].

The statements about the converse and the transitive closure modalities are due to [Kikot & Zolin & Sh, 2014].

A class of frames $\mathcal{F}$ *admits filtration* if, for any finite Sub-closed set of formulas Γ and an $\mathcal{F}$-model M, there exists a finite $\mathcal{F}$-model that is a Γ-filtration of M.
A class of models $\mathcal{M}$ *admits filtration* if, for any finite Sub-closed set of formulas Γ and any $M \in \mathcal{M}$, there is a finite model in $\mathcal{M}$ that is a Γ-filtration of M.

Various versions of the AF property:
[Goranko & Passy 1992]
[G. Bezhanishvili & Zakharyaschev, 1997]
[N. Bezhanishvili & Ten Cate, 2006]
[Schmidt & Tishkovsky, 2008]
[Ilin, 2018]
...

If L is Kripke complete and the class of its frames Frames(L) admits filtration, then L has the FMP.

If the class of models Mod(L) of a logic L admits filtration, then L has the FMP and hence is Kripke complete.

Theorem (Filtration safe operations on frames)

Suppose that a class $\mathcal{F}$ of frames admits filtration. Then the classes $\mathcal{F}^u$, $\mathcal{F}^t$, $\mathcal{F}^+$ admit filtrations too.

Corollary

Suppose that the class of frames of a logic L admits filtration. Then L^u, L^t, L^+ have the fmp provided that they are Kripke complete.

The statements about the universal modality are due to [Goranko & Passy, 1992].

The statements about the converse and the transitive closure modalities are due to [Kikot & Zolin & Sh, 2014].

If a logic L is canonical, then L^t and L^u are canonical (so Kripke complete).
This is not the case for L^+.

A class of frames $\mathcal{F}$ *admits filtration* if, for any finite Sub-closed set of formulas Γ and an $\mathcal{F}$-model M, there exists a finite $\mathcal{F}$-model that is a Γ-filtration of M.
A class of models $\mathcal{M}$ *admits filtration* if, for any finite Sub-closed set of formulas Γ and any $M \in \mathcal{M}$, there is a finite model in $\mathcal{M}$ that is a Γ-filtration of M.

Various versions of the AF property:
[Goranko & Passy 1992]
[G. Bezhanishvili & Zakharyaschev, 1997]
[N. Bezhanishvili & Ten Cate, 2006]
[Schmidt & Tishkovsky, 2008]
[Ilin, 2018]
...

If L is Kripke complete and the class of its frames Frames(L) admits filtration, then L has the FMP.

If the class of models Mod(L) of a logic L admits filtration, then L has the FMP and hence is Kripke complete.

Theorem (Filtration safe operations on frames)

Suppose that a class $\mathcal{F}$ of frames admits filtration. Then the classes $\mathcal{F}^u$, $\mathcal{F}^t$, $\mathcal{F}^+$ admit filtrations too.

Corollary

Suppose that the class of frames of a logic L admits filtration. Then L^u, L^t, L^+ have the fmp provided that they are Kripke complete.

The statements about the universal modality are due to [Goranko & Passy, 1992].

The statements about the converse and the transitive closure modalities are due to [Kikot & Zolin & Sh, 2014].

If a logic L is canonical, then L^t and L^u are canonical (so Kripke complete).
This is not the case for L^+.

There is a semantic condition of L sufficient for the Kripke completeness of L^+.

A class of frames $\mathcal{F}$ *admits filtration* if, for any finite Sub-closed set of formulas Γ and an $\mathcal{F}$-model M, there exists a finite $\mathcal{F}$-model that is a Γ-filtration of M.
A class of models $\mathcal{M}$ *admits filtration* if, for any finite Sub-closed set of formulas Γ and any $M \in \mathcal{M}$, there is a finite model in $\mathcal{M}$ that is a Γ-filtration of M.

Various versions of the AF property:
[Goranko & Passy 1992]
[G. Bezhanishvili & Zakharyaschev, 1997]
[N. Bezhanishvili & Ten Cate, 2006]
[Schmidt & Tishkovsky, 2008]
[Ilin, 2018]
...

If L is Kripke complete and the class of its frames Frames(L) admits filtration, then L has the FMP.

If the class of models Mod(L) of a logic L admits filtration, then L has the FMP and hence is Kripke complete.

A Γ-filtration $\widehat{M} = (W/{\sim}, \ldots)$ is *definable* if $\sim = \sim_\Psi$ for some set of formulas $\Psi \supseteq \Gamma$.

Theorem ([Zolin, Sh, 2015])

If the class Mod(L) *admits definable filtration, then so does the class* Mod(L^+).

A class of frames $\mathcal{F}$ *admits filtration* if, for any finite Sub-closed set of formulas Γ and an $\mathcal{F}$-model M, there exists a finite $\mathcal{F}$-model that is a Γ-filtration of M.
A class of models $\mathcal{M}$ *admits filtration* if, for any finite Sub-closed set of formulas Γ and any $M \in \mathcal{M}$, there is a finite model in $\mathcal{M}$ that is a Γ-filtration of M.

Various versions of the AF property:
[Goranko & Passy 1992]
[G. Bezhanishvili & Zakharyaschev, 1997]
[N. Bezhanishvili & Ten Cate, 2006]
[Schmidt & Tishkovsky, 2008]
[Ilin, 2018]
. . .

If L is Kripke complete and the class of its frames Frames(L) admits filtration, then L has the FMP.

If the class of models Mod(L) of a logic L admits filtration, then L has the FMP and hence is Kripke complete.

A Γ-filtration $\widehat{M} = (W/{\sim}, \ldots)$ is *definable* if $\sim = \sim_\Psi$ for some set of formulas $\Psi \supseteq \Gamma$.

Theorem ([Zolin, Sh, 2015])

If the class Mod(L) *admits definable filtration, then so does the class* $\mathrm{Mod}(\mathrm{L}^+)$.

For any logic L, if Mod(L) admits (definable) filtration, then so does Frames(L).

Proposition (ADF for frames implies ADF for models)

If L is a canonical logic, then Frames(L) *admits definable filtration iff so does* Mod(L).

For an alphabet Σ, let $\Sigma^\sharp = \Sigma \cup \{(e \circ f), (e \cup f), e^+ \mid e, f \in \Sigma\}$, assuming that the added symbols are not in Σ. Put $\Sigma^{(0)} = \Sigma$, $\Sigma^{(n+1)} = (\Sigma^{(n)})^\sharp$.
For a frame $F = (W, (R_e)_{e\in\Sigma})$, put $F^\sharp = (W, (R_e)_{e\in\Sigma^\sharp})$, where for $e, c \in \Sigma$,

$$R_{e\circ c} = R_e \circ R_c, \quad R_{e\cup c} = R_e \cup R_c, \quad R_{e^+} = (R_e)^+.$$

Put $F^{(0)} = F$, $F^{(n+1)} = (F^{(n)})^\sharp$.
For a logic L over Σ, let $L^\sharp$ be the smallest (normal) logic over $\Sigma^\sharp$ that contains L and the following PDL-like axioms, for all $e, c \in \Sigma$:

$$[e \cup c]p \leftrightarrow [e]p \wedge [c]p,$$
$$[e \circ c]p \leftrightarrow [e][c]p,$$
$$[e^+]p \to [e]p, \quad [e^+]p \to [e][e^+]p, \quad [e^+](p \to [e]p) \to ([e]p \to [e^+]p).$$

We put $L^{(0)} = L$, $L^{(n+1)} = (L^{(n)})^\sharp$.

Corollary ([Kikot, Zolin, Sh, 2020])

Let L be a logic over a finite alphabet Σ. If the class of its models Mod(L) *admits definable filtration, then, for every $n < \omega$, we have:*

1. Mod($L^{(n)}$) *admits definable filtration.*
2. *$L^{(n)}$ has the finite model property; a fortiori, $L^{(n)}$ is Kripke complete.*
3. *If L is finitely axiomatizable, then $L^{(n)}$ is decidable.*

Example

Let each $L_1, \ldots, L_k$ be any of the logics $\mathrm{K}, \mathrm{T}, \mathrm{B}, \mathrm{K4}, \mathrm{S4}, \mathrm{S5}$. Then, for any $n < \omega$, the logic $(L_1 * \ldots * L_k)^{(n)}$ has the fmp and is decidable.

That the class Mod(L) admits definable filtration is sufficient for the Kripke completeness of L^+.

Problem

Syntactic condition(s) on L for the Kripke completeness of L^+.

[Kikot, 2015] Sufficient firs-order conditions on admits definable filtrations (in some strict sense: $\sim = \sim_\Gamma$).

[Ilin, 2016] A family of extensions of PDL with the FMP.

That the class Mod(L) admits definable filtration is sufficient for the Kripke completeness of L^{+}.

Problem

Syntactic condition(s) on L for the Kripke completeness of L^{+}.

[Kikot, 2015] Sufficient firs-order conditions on admits definable filtrations (in some strict sense: $\sim=\sim_{\Gamma}$).

[Ilin, 2016] A family of extensions of PDL with the FMP.

Question

Is there a transitive canonical logic L which has the FMP, but does not admit filtration?

Negative answer might be useful for negative results about decidability.

An algebra A is *locally finite* if every finitely generated subalgebra of A is finite.

A logic L is *locally finite* (or *locally tabular*) if for all $k < \omega$ there are only finitely many k-formulas (i.e., formulas in k variables) up to $\leftrightarrow_{\mathrm{L}}$.

TFAE:

L is locally finite.	Every finitely generated Lindenbaum-Tarski (i.e., free) algebra of L is finite.	The variety of L-algebras is *locally finite*, i.e., every finitely generated L-algebra is finite.

$Log(F)$ is LF $\underset{\not\Leftarrow}{\Rightarrow}$ $Alg(F)$ is LF $\underset{\not\Leftarrow}{\Rightarrow}$ $Log(F)$ has the FMP

- Segerberg, K., "An Essay in Classical Modal Logic," 1971.
- Kuznetsov, A. *Some properties of the structure of varieties of pseudo-Boolean algebras*, 1971.
- Maksimova, L. *Modal logics of finite slices*, 1975.
- Komori, Y. *The finite model property of the intermediate propositional logics on finite slices*, 1975.
- Mardaev, S. *The number of prelocally tabular superintuitionistic propositional logics*, 1984.
- Citkin, A. *Finite axiomatizability of locally tabular superintuitionistic logics*, 1986.
- ...
- Bezhanishvili, G. *Varieties of monadic Heyting algebras. part I*, 1998.
- Bezhanishvili, G. and Grigolia, R. *Locally tabular extensions of MIPC*, 1998.
- Bezhanishvili, G., *Locally finite varieties*, 2001.
- Bezhanishvili, N. *Varieties of two-dimensional cylindric algebras. part I: Diagonal-free case*, 2002.
- Shehtman, V. *Canonical filtrations and local tabularity*, 2014.
- Shapirovsky, I. and Shehtman, V. *Local tabularity without transitivity*, 2016
- ...

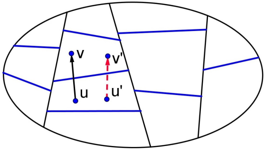
[Shehtman, 2014] If L is locally finite, then it admits definable filtration.

Let $F = (W, R)$ be a frame. A partition $\mathcal{A}$ of W is *tuned* if for every $U, V \in \mathcal{A}$,

$$\exists u \in U\ \exists v \in V\ uRv \quad \Rightarrow \quad \forall u \in U\ \exists v \in V\ uRv.$$

F is said to be *tunable* if every finite partition $\mathcal{A}$ of F admits a finite tuned refinement.

The key tool: The algebra of F is locally finite iff F is tunable.

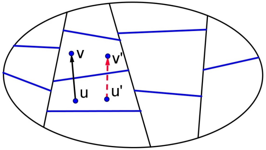


Let $F = (W, R)$ be a frame. A partition $\mathcal{A}$ of W is *tuned* if for every $U, V \in \mathcal{A}$,

$$\exists u \in U\ \exists v \in V\ uRv \quad \Rightarrow \quad \forall u \in U\ \exists v \in V\ uRv.$$

F is said to be *tunable* if every finite partition $\mathcal{A}$ of F admits a finite tuned refinement.

The key tool: The algebra of F is locally finite iff F is tunable.



TFAE:

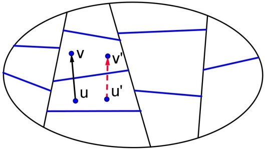
- $\mathcal{A}$ is tuned in F
- The equivalence $\sim$ defined by $\mathcal{A} = W/\sim$ satisfies the condition
  $$\sim \circ R \ \subseteq \ R \circ \sim,$$
  i.e., $\sim$ is a bisimulation w.r.t. R on W.
- $x \mapsto [x]_{\mathcal{A}}$ is a p-morphism from F onto the "Franzen's filtration" $(\mathcal{A}, R_{\mathcal{A}})$, where for $U, V \in \mathcal{A}$,
  $$UR_{\mathcal{A}}V \text{ iff } \exists u \in U\ \exists v \in V\ uRv$$
  [Segerberg, K.: Franzen's proof of Bull's theorem. Ajatus 35, 216–221 (1973)]
- Unions of elements of $\mathcal{A}$ form a subalgebra of $Alg(F)$.

Let $F = (W, R)$ be a frame. A partition $\mathcal{A}$ of W is *tuned* if for every $U, V \in \mathcal{A}$,

$$\exists u \in U\, \exists v \in V\; uRv \quad \Rightarrow \quad \forall u \in U\, \exists v \in V\; uRv.$$

F is said to be *tunable* if every finite partition $\mathcal{A}$ of F admits a finite tuned refinement.

The key tool: The algebra of F is locally finite iff F is tunable.



Theorem [Malcev, 1960s]

The variety $\mathrm{Var}(A)$ of a finite signature is LF iff there exists $f : \omega \to \omega$ s.t. the cardinality of a subalgebra of A generated by $m < \omega$ elements is $\leq f(m)$.

Corollary [Shehtman & Sh, 2016]

$Log(F)$ is LF iff there exists $f : \omega \to \omega$ s.t. every finite partition $\mathcal{A}$ of F admits a tuned finite refinement $\mathcal{B}$ with $|\mathcal{B}| \leq f(|\mathcal{A}|)$.

TFAE:

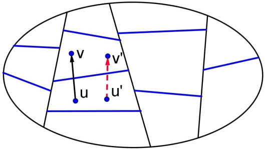
- $\mathcal{A}$ is tuned in F
- The equivalence $\sim$ defined by $\mathcal{A} = W/{\sim}$ satisfies the condition
$$\sim \circ R \;\subseteq\; R \circ \sim,$$
i.e., $\sim$ is a bisimulation w.r.t. R on W.
- $x \mapsto [x]_{\mathcal{A}}$ is a p-morphism from F onto the "Franzen's filtration" $(\mathcal{A}, R_{\mathcal{A}})$, where for $U, V \in \mathcal{A}$,
$$UR_{\mathcal{A}}V \text{ iff } \exists u \in U\, \exists v \in V\; uRv$$
[Segerberg, K.: Franzen's proof of Bull's theorem. Ajatus 35, 216–221 (1973)]
- Unions of elements of $\mathcal{A}$ form a subalgebra of $Alg(F)$.

Let $F = (W, R)$ be a frame. A partition $\mathcal{A}$ of W is *tuned* if for every $U, V \in \mathcal{A}$,

$$\exists u \in U \, \exists v \in V \; uRv \quad \Rightarrow \quad \forall u \in U \, \exists v \in V \; uRv.$$

F is said to be *tunable* if every finite partition $\mathcal{A}$ of F admits a finite tuned refinement.

The key tool: The algebra of F is locally finite iff F is tunable.



Theorem [Malcev, 1960s]

The variety $\mathrm{Var}(A)$ of a finite signature is LF iff there exists $f : \omega \to \omega$ s.t. the cardinality of a subalgebra of A generated by $m < \omega$ elements is $\leq f(m)$.

Corollary [Shehtman & Sh, 2016]

$Log(F)$ is LF iff there exists $f : \omega \to \omega$ s.t. every finite partition $\mathcal{A}$ of F admits a tuned finite refinement $\mathcal{B}$ with $|\mathcal{B}| \leq f(|\mathcal{A}|)$.

[Segerberg, 1971; Maksimova, 1975] A transitive logic L is locally finite iff L is of finite height.

The non-transitive case is much more complicated and less investigated.
[Shehtman, Sh, 2016] This criterion holds for all logics containing $\Diamond^m p \to \Diamond p \vee p$, $m > 1$.

L is *pretransitive* if there is a formula $\Diamond^*(p)$ ('master modality') s.t. $\Diamond^*(\varphi)$ expresses the satisfiability of φ in cones on models of L.

Pretransitive examples:

$\mathrm{K4}, \mathrm{wK4} = [\Diamond\Diamond p \to \Diamond p \vee p]$, $\mathrm{K5} = [\Diamond p \to \Box\Diamond p]$, $[\Diamond^n p \to \Diamond^m p]$ for $n > m$, products of transitive logics

Shehtman, Sh, 2016: Every 1-finite (a fortiori, locally finite) modal logic is a pretransitive logic of finite height.

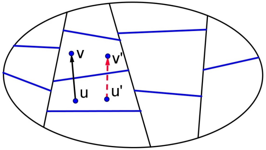
Makinson, 1981: In general, the converse is not true.
There exists a pretransitive L s.t. $\mathrm{L}[1]$, the extension of L with the axiom of height 1, is not 1-finite. (Put $\mathrm{L} = [\Diamond^3 p \to \Diamond^2 p]$.)

Let $F = (W, R)$ be a frame. A partition $\mathcal{A}$ of W is *tuned* if for every $U, V \in \mathcal{A}$,

$$\exists u \in U \, \exists v \in V \; uRv \quad \Rightarrow \quad \forall u \in U \, \exists v \in V \; uRv.$$

F is said to be *tunable* if every finite partition $\mathcal{A}$ of F admits a finite tuned refinement.

The key tool: The algebra of F is locally finite iff F is tunable.



Theorem [Malcev, 1960s]

The variety $\mathrm{Var}(A)$ of a finite signature is LF iff there exists $f : \omega \to \omega$ s.t. the cardinality of a subalgebra of A generated by $m < \omega$ elements is $\leq f(m)$.

Corollary [Shehtman & Sh, 2016]

$Log(F)$ is LF iff there exists $f : \omega \to \omega$ s.t. every finite partition $\mathcal{A}$ of F admits a tuned finite refinement $\mathcal{B}$ with $|\mathcal{B}| \leq f(|\mathcal{A}|)$.

A logic is said to be *k-finite* if, up to the provable equivalence, there exist only finitely many k-formulas.
[Maksimova, 1975]
A transitive logic is locally finite iff it is 1-finite.

Strange fact. If the logic (algebra) of a frame F is locally finite, then the logic (algebra) of any subframe of F is also locally finite.

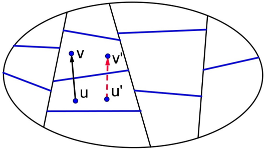
Corollary. There exists a unimodal 1-finite logic which is not locally finite:
There is a frame F s.t. every 2-element partition can be tuned, while F contains a subframe of infinite hight.

Let $F = (W, R)$ be a frame. A partition $\mathcal{A}$ of W is *tuned* if for every $U, V \in \mathcal{A}$,

$$\exists u \in U \, \exists v \in V \; uRv \quad \Rightarrow \quad \forall u \in U \, \exists v \in V \; uRv.$$

F is said to be *tunable* if every finite partition $\mathcal{A}$ of F admits a finite tuned refinement.

The key tool: The algebra of F is locally finite iff F is tunable.



Theorem [Malcev, 1960s]

The variety $\mathrm{Var}(A)$ of a finite signature is LF iff there exists $f : \omega \to \omega$ s.t. the cardinality of a subalgebra of A generated by $m < \omega$ elements is $\leq f(m)$.

Corollary [Shehtman & Sh, 2016]

$Log(F)$ is LF iff there exists $f : \omega \to \omega$ s.t. every finite partition $\mathcal{A}$ of F admits a tuned finite refinement $\mathcal{B}$ with $|\mathcal{B}| \leq f(|\mathcal{A}|)$.

It is unknown whether 2-finiteness of a modal logic implies local finiteness.

Chagrov's modal formulas correspond to

$$\forall x_0, \ldots, x_{m+1} \left(x_0 R x_1 \ldots R x_{m+1} \to \bigvee_{i<j} x_i = x_j \right).$$

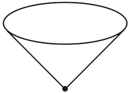
[Chagrov, Shehtman, 1994] Logics containing Chagrov's formulas are LF.

Consider the following first-order properties P_m:

$$\forall x_0, \ldots, x_{m+1} \left(x_0 R x_1 \ldots R x_{m+1} \to \bigvee_{i<j} x_i = x_j \vee \bigvee_{i+1<j} x_i R x_j \right).$$

Observation. If the logic of F is 2-finite, then P_m holds in F for some m.

Final remark: Franzen's filtrations might be very useful to prove the FMP when the axiomatization is unknown.



For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\mu(x) = -x_n^2 + \sum_{i=1}^{n-1} x_i^2$
Chronological $\prec$ and *causal* $\preceq$ *future*:
$x \prec y \;\rightleftharpoons\; \mu(x-y) < 0 \;\&\; x_n < y_n$
$x \preceq y \;\rightleftharpoons\; \mu(x-y) \leq 0 \;\&\; x_n \leq y_n$
Goldblatt, 1980; Shehtman, 1983: For $n \geq 2$, the modal logic of $(\mathbb{R}^n, \preceq)$ is $\mathrm{S4.2} = [\Diamond\Diamond p \to \Diamond p,\ p \to \Diamond p,\ \Diamond\Box p \to \Box\Diamond p]$.
Problems of Goldblatt:

1. Axiomatize the logics corresponding to $\prec$ in the various dimensions.
2. Axiomatize the *bimodal* logics of $(\mathbb{R}^n, \preceq, \succeq)$ and of $(\mathbb{R}^n, \prec, \succ)$.
3. Analyze the logic of *discrete spacetime*.

Problems 1 and 3 were formulated in 1980, Problem 2 in 1992.
Solutions and partial solutions:

1. Shehtman & Sh, 2002: Finite axiomatization and the FMP of the logic of $\prec$ (all dimensions).
2. Hirsch & Reynolds, 2018: The logic of $(\mathbb{R}^2, \preceq, \succeq)$ is decidable (in PSPACE).
   Hirsch & McLean, 2018: The logic of $(\mathbb{R}^2, \prec, \succ)$ is decidable (in PSPACE).
3. Sh, 2019: $(\mathbb{Z}^2, \prec)$ and $(\mathbb{Z}^2, \preceq)$ have logics with the FMP
   (Explanation: the direct squares $(\omega, <)^2$, $(\omega, \leq)^2$ are tunable).

In the 2-dimensional case, the above structures are direct squares of linear orders.

Question

Let frames F_1 and F_2 be tunable. Is the direct product $F_1 \times F_2$ tunable?

In the other words:
if $Alg(F_1)$ and $Alg(F_2)$ are LF, is the algebra $Alg(F_1 \times F_2)$ LF?

Thank you!