Decidability of modal logics of non-k-colorable graphs

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Modal language

The set of *n-modal formulas* is built from a countable set of propositional variables $\mathrm{PV} = \{p_0, p_1, \ldots\}$ using Boolean connectives and unary connectives $\Diamond_i$, $i < n$ (*modalities*).

Kripke semantics

An *n-frame* F: $(X, (R_i)_{i<n})$, where R_i are binary relations on a set X.
A *model* M on F is a pair (F, θ) where $\theta : \mathrm{VAR} \to \mathcal{P}(X)$.

$$M, x \vDash p \text{ iff } x \in \theta(p), \qquad M, x \vDash \Diamond_i \varphi \text{ iff } M, y \vDash \varphi \text{ for some } y \text{ with } xR_i y.$$

A formula φ is *true in a model* M, in symbols $M \vDash \varphi$, if $M, x \vDash \varphi$ for all x in M.
A formula φ is *valid in a frame* F, in symbols $F \vDash \varphi$, if φ is true in every model on F.

Examples (Unimodal case)

$(X, R) \vDash p \to \Diamond p \iff R$ is reflexive;

$(X, R) \vDash p \to \Box\Diamond p \iff R$ is symmetric ($\Box\varphi$ denotes $\neg\Diamond\neg\varphi$);

$(X, R) \vDash \Diamond\top \iff \forall x \exists y\, xRy$;

$(X, R) \vDash \Diamond p \to \Diamond(p \wedge \neg\Diamond p) \iff (X, R^{-1})$ is a well-founded strict poset.

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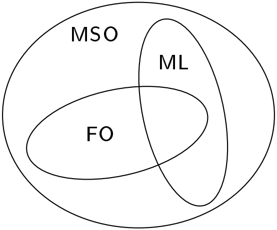

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Example (Bimodal case)

Consider a structure $(X, R, X \times X)$, R is symmetric.

R interprets $\Diamond_0$, the universal relation $X \times X$ interprets $\Diamond_1$.

We have:
$$(X, R, X \times X) \vDash \Diamond_1 p \wedge \Diamond_1 \neg p \to \Diamond_1 (p \wedge \Diamond_0 \neg p) \iff$$

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$$(X, R, X \times X) \vDash \Diamond_1 p \wedge \Diamond_1 \neg p \to \Diamond_1(p \wedge \Diamond_0 \neg p) \iff (X, R) \text{ is connected.}$$

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As usual, a partition $\mathcal{A}$ of a set X is a family of non-empty pairwise disjoint sets such that $X = \bigcup \mathcal{A}$.

Definition

Let X be a set, $R \subseteq X \times X$. A partition (in other terms: coloring) $\mathcal{A}$ of X is *proper*, if

$$\forall A \in \mathcal{A}\, \forall x \in A\, \forall y \in A\, \neg xRy.$$

The *chromatic number* $\chi(X, R)$ *of* (X, R) is the least k in the set

$$\{|\mathcal{A}| : \mathcal{A} \text{ is a finite proper partition of } X\}$$

(if the set is empty, $\chi(X, R) = \infty$.)

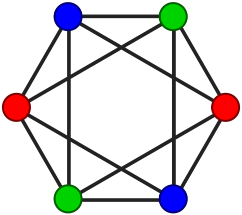

Figure: Wikipedia/Graph coloring

For a unimodal frame $F = (X, R)$, let $F_{\neq}$ be the bimodal frame $(X, R, \neq_X)$, where $\neq_X$ is the inequality relation on X, i.e., the set of pairs $(x, y) \in X \times X$ such that $x \neq y$.

From now on, we write $\Diamond$ for $\Diamond_0$, and $\langle\neq\rangle$ for $\Diamond_1$; likewise for boxes.
We also use abbreviations $\exists\varphi$ for $\langle\neq\rangle\varphi \vee \varphi$ and $\forall\varphi$ for $[\neq]\varphi \wedge \varphi$.

Put

$$\chi_k^{>} = \forall \bigvee_{i<k} (p_i \wedge \bigwedge_{i \neq j < k} \neg p_j) \to \exists \bigvee_{i<k} (p_i \wedge \Diamond p_i).$$

Proposition (Follows from [Hughes 1990])

The chromatic number of $F > k$ iff $F_{\neq} \vDash \chi_k^{>}$.

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Proposition (Follows from [Hughes 1990])

The chromatic number of $F > k$ *iff* $F_{\neq} \models \chi_k^{>}$.

Historical remark

In [Goldblatt, Hodkinson, Venema 2004], these formulas were used to construct a *canonical logic* which cannot be determined by a first-order definable class of relational structures; this gave a solution of a long-standing problem [Fine 1975].

For a class $\mathcal{C}$ of frames, the set $\mathrm{Log}\,\mathcal{C} = \{\varphi \mid \mathcal{C} \vDash \varphi\}$ is called the *logic of $\mathcal{C}$*.

General problems

- complete axiomatization of $\mathrm{Log}\,\mathcal{C}$;
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Definitions

A set L of formulas is a *modal logic* (in a more accurate terminology — *normal propositional modal logic*), if L contains the classical tautologies, the formulas

$$\Diamond_i \bot \leftrightarrow \bot, \quad \Diamond_i(p \vee q) \leftrightarrow \Diamond_i p \vee \Diamond_i q \quad (i < n),$$

and is closed under the rules of MP, substitution and *monotonicity*: if $(\varphi \to \psi) \in L$, then $(\Diamond_i \varphi \to \Diamond_i \psi) \in L$.

A logic L is *Kripke complete*, if L is the logic of a class $\mathcal{C}$ of Kripke frames: $L = \mathrm{Log}\,\mathcal{C}$.

A logic L has the *finite model property*, if L is the logic of a class $\mathcal{C}$ of finite frames.

Fact

If L has the fmp and is finitely axiomatizable, then it is decidable.

K is the least unimodal logic. KB is the least unimodal logic that contains the formula $p \to \Box\Diamond p$ (recall: the formula expresses symmetry of relation).

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For a unimodal logic L, let $L_{\neq}$ be the smallest bimodal logic that contains L and

$$p \to [\neq]\langle\neq\rangle p, \quad \langle\neq\rangle\langle\neq\rangle p \to \exists p, \quad \Diamond p \to \exists p.$$

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Theorem

1. Let $\mathcal{G}^{>k}$ be the class of graphs G such that $\chi(G) > k$, and let $\mathcal{D}^{>k}$ be the class of directed graphs G such that $\chi(G) > k$. Then

$$\mathrm{Log}\,\mathcal{G}^{>k}_{\neq} \text{ is } \mathrm{KB}_{\neq} + \chi^{>}_{k}, \text{ and } \mathrm{Log}\,\mathcal{D}^{>k}_{\neq} \text{ is } \mathrm{K}_{\neq} + \chi^{>}_{k}.$$

2. For each $k < \omega$, the logics $\mathrm{KB}_{\neq} + \chi^{>}_{k}$ and $\mathrm{K}_{\neq} + \chi^{>}_{k}$ have the exponential finite model property and are decidable.

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Update: A related result was obtained very recently in [Ding, Liu & Wang, 2023]: it was shown that in *neighborhood semantics* of modal language, the non-k-colorability of *hypergraphs* is expressible, and the resulting modal systems are decidable as well.

I am grateful to Gillman Payette for sharing with me this reference after my talk at WoLLIC.

A frame $F = (X, R)$ is *connected*, if for any points x, y in X, there are points $x_0 = x, x_1, \ldots, x_n = y$ such that for each $i < n$, $x_i R x_{i+1}$ or $x_{i+1} R x_i$.
Let $\textsc{Con}$ be the following formula:

$$\exists p \wedge \exists \neg p \to \exists (p \wedge \Diamond \neg p). \tag{1}$$

Recall: for every graph G,

$$G \text{ is connected iff } G_{\neq} \vDash \textsc{Con}.$$

Theorem

1. Let $\mathcal{C}^{>k}$ be the class of non-k-colorable connected non-singleton graphs. Then

$$\mathrm{Log}\, \mathcal{C}^{>k}_{\neq} \textit{ is } \mathrm{KB}_{\neq} + \{\chi^{>}_{k}, \textsc{Con}, \Diamond \top\}.$$

2. All logics $\mathrm{KB}_{\neq} + \{\chi^{>}_{k}, \textsc{Con}, \Diamond \top\}$ have the exponential finite model property and are decidable.

normal modal logics $\supsetneq$
Kripke complete logics $\supsetneq$
logics with the finite model property $\supsetneq$
logics that *admit filtration*

Informally, filtration is a method of collapsing an infinite model into a finite one while preserving the truth value of a given formula. It is widely used for establishing the finite model property and decidability of modal logics.
A logic L *admits filtration* iff any L-model can be "filtrated" into a finite L-model.

Formally:

For a model $M = (X, (R_i)_{i<n}, \theta)$ and a set Γ of formulas, put

$$x \sim_\Gamma y \text{ iff } \forall \psi \in \Gamma \ (M, x \models \psi \text{ iff } M, y \models \psi).$$

A *Γ-filtration* of M is a model $\widehat{M} = (\widehat{X}, (\widehat{R}_i)_{i<n}, \widehat{\theta})$ such that:

$\widehat{X} = X/{\sim}$ for some equivalence relation $\sim$ finer than $\sim_\Gamma$;

$\widehat{M}, [x] \models p$ iff $M, x \models p$ for all $p \in \Gamma$.

For all $i < n$, we have $(R_i)_\sim \subseteq \widehat{R}_i \subseteq (R_i)^\Gamma_\sim$, where

$$[x]\,(R_i)_\sim\,[y] \quad \text{iff} \quad \exists x' \sim x\ \exists y' \sim y\ (x'\,R_i\,y'),$$
$$[x]\,(R_i)^\Gamma_\sim\,[y] \quad \text{iff} \quad \forall \psi\ (\Diamond_i \psi \in \Gamma\ \&\ M, y \models \psi \Rightarrow M, x \models \Diamond_i \psi).$$

If $\sim = \sim_\Psi$ for some finite set of formulas $\Psi \supseteq \Gamma$, then $\widehat{M}$ is called a *definable Γ-filtration* of M.

A logic L *admits (rooted) definable filtration*, if for any (point-generated) model M with $M \models L$, and for any finite subformula-closed set of formulas Γ, there exists a finite model $\widehat{M}$ with $\widehat{M} \models L$ that is a definable Γ-filtration of M.

It is well-known that many standard logics admit filtration and hence have the finite model property.
Moreover, in many cases filtrability of a logic leads to the finite model property of reacher systems.
For example, if a modal logic L admits definable filtration, then its enrichments with modalities for the transitive closure and converse relations also admit definable filtration (that is, you can build a PDL extension of such an L and keep the finite model property) [Kikot, Zolin, Sh, 2014; 2020].

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Theorem

If a bimodal logic L admits definable filtration, then all $L + \chi_k^{>}$ admit definable filtration, and consequently have the finite model property.

Theorem

Assume that a bimodal logic L admits rooted definable filtration, $k < \omega$. Then $L + \chi_k^{>}$ has the finite model property. If also L extends $\mathrm{KB}_{\neq}$, then $L + \{\chi_k^{>}, \textsc{Con}\}$ has the finite model property.

Modal logics of different classes of non-k-colorable graphs are decidable. It is of definite interest to consider logics of certain graphs, for which the chromatic number is unknown.

Let $F = (\mathbb{R}^2, R_{=1})$ be the unit distance graph of the real plane.

Hadwiger–Nelson problem (1950s)

What is $\chi(F)$?

It is known that $5 \leq \chi(F) \leq 7$ ([≤ 7: Isbell, 1950s]; [$5 \leq$: Aubrey De Grey, 2018]).

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In modal terms, the Hadwiger–Nelson problem asks whether $\chi_5^>, \chi_6^>$ belong to $L_{=1}$.

We know that $L_{=1}$ extends $L = \mathrm{KB}_{\neq} + \{\chi_4^>, \mathrm{Con}, \Diamond\top, \Diamond p \to \langle\neq\rangle p\}$ (the latter logic is decidable). However, $L_{=1}$ contains extra formulas. For example, let

$$\mathrm{P}(k,m,n) = \bigwedge_{i<k} \Diamond^m \Box^n p_i \to \bigvee_{i \neq j < k} \Diamond^m (p_i \wedge p_j).$$

For various k, m, n, $\mathrm{P}(k,m,n)$ is in $L_{=1}$ (and not in L); this can be obtained from known solutions for problems of packing equal circles in a circle.

Problem

Is $L_{=1}$ decidable? Finitely axiomatizable? Recursively enumerable? Does it have the finite model property?

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Let $V_r \subseteq \mathbb{R}^2$ be a disk of radius r. It follows from de Bruijn–Erdős theorem that if $\chi(F) > k$, then $\chi(V_r, R_{=1}) > k$ for some r.

Let $L_{=1,r}$ be the unimodal logic of the frame $F_r = (V_r, R_{=1})$ $(r > 1)$. In this case, the properties

$$\chi(F) > k$$

are expressible in the unimodal language.

Problem

To analyze the unimodal logics $L_{=1,r}$.

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