On the finite model property of subframe pretransitive logics

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Language: a countable set of variables VAR, Boolean connectives, and unary $\Diamond$'s

Normal modal logic

A set of modal $\mathcal{A}$-formulas L is a *normal modal logic*, if for all diamonds $\Diamond$ in the language, L contains

classical tautologies,

$\Diamond\bot \leftrightarrow \bot, \quad \Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q,$

and is closed under MP, Sub, and *Mon*: if $(\varphi \to \psi) \in L$, then $(\Diamond\varphi \to \Diamond\psi) \in L$.

Kripke semantics

A *(Kripke) frame* F is a set X with binary relation(s) $R_\Diamond$.
A *model* M on F is a pair (F, θ) where $\theta : \mathrm{VAR} \to \mathcal{P}(X)$.

$M, x \vDash p$ iff $x \in \theta(p)$, $\quad M, x \vDash \Diamond\varphi$ iff $M, y \vDash \varphi$ for some y with $xR_\Diamond y$.

$\mathrm{Log}(\mathsf{F}) = \{\varphi \mid \mathsf{F} \vDash \varphi\}$, where $\mathsf{F} \vDash \varphi$ means that $M, x \vDash \varphi$ for all M on F and all x in M.

A logic L is *Kripke complete*, if L is the logic of a class $\mathcal{C}$ of Kripke frames:
$L = \bigcap\{\mathrm{Log}(\mathsf{F}) \mid \mathsf{F} \in \mathcal{C}\}$.

A logic L has the *finite model property*, if L is the logic of a class $\mathcal{C}$ of finite frames (algebras, models).

The *tense expansion* L_{t} of a Kripke-complete logic L is the bimodal logic of the class $\{(X, R, R^{-1}) \mid (X, R) \vDash L\}$.

We are interested in modal logics with expressible transitive reflexive closure modality. Such logics are said to be *pretransitive*. In terms of frames, this means that for some fixed finite m, in frames of the logic we have

$$R^{m+1} \subseteq \bigcup_{i \leq m} R^i,$$

where R^0 is the diagonal, $R^{i+1} = R \circ R^i$, and $\circ$ is the composition. We address this property as *m-transitivity*.

General problem

The finite model property of pretransitive logics and their tense expansions.

Numerous results are known for the case $m = 1$, only a few for $m > 1$...

m-transitivity: $\quad R^{m+1} \subseteq \bigcup_{i \leq m} R^i$

$m = 1$: $R^2 \subseteq R^0 \cup R$

[McKinsey, 1941] FMP of the logic $\mathrm{S4}$ of all preorders $R^2 \cup R^0 \subseteq R$.
[Lemmon, 1966] FMP of the logic $\mathrm{K4}$ of all transitive relations $R^2 \subseteq R$.
[Segerberg, 1970] FMP of the tense expansions (another diamond for the inverse relation) of $\mathrm{K4}$, $\mathrm{S4}$.
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$m > 1$

For $m > 1$, the FMP of the logic of all m-transitive relations is a Big Open Problem.

FMP is known for the following subclasses of m-transitive frames:
[Gabbay, 1972] $R^{m+1} \subseteq R$ (m-transitive analogs of $\mathrm{K4}$);
[Kudinov & Sh, 2015] FMP of the logics of m-transitive relations with bounded height of their skeletons;
[Kudinov & Sh, 2023] FMP of the logics L_m of relations with $R^{m+1} \subseteq R^0 \cup R$ (m-transitive analogs of $\mathrm{wK4}$) and for tense expansions of L_m; FMP of subframe extensions of L_m...
[Dvorkin, 2024] ... and actually for subframe m-transitive logics weaker than L_m.

<u>Remark.</u> Except for logics in [Kudinov & Sh, 2015], the logics are subframe.

Strategy

Given a model $M \vDash L$ and a formula φ satisfiable in M, we want to construct a finite model $\widehat{M}$ such that

$\widehat{M} \vDash L$, and

φ is satisfiable in $\widehat{M}$.

Two standard approaches

Filtrations in the style of Lemmon-Scott-Segerberg, or *epi*-filtrations:

$\widehat{M}$ is a quotient (of special kind) of M.

Selective filtrations:

$\widehat{M}$ is a submodel (of special kind) of M.

Transfer results

In many cases, the proof of the FMP of L results in the FMP for other systems.

In the case of selective filtrations, one might expect FMP for subframe extensions of L [Fine, 1985].

In the case of epi-filtrations, FMP transfers to the tense (and some other) expansion of L [Kikot & Zolin & Sh, 2014].

The idea of epifiltrations

Any set of formulas Γ on a model M induces an equivalence $\sim_\Gamma$ on M. Any equivalence $\sim$ finer than $\sim_\Gamma$ induces two relations: the *minimal filtered relation* $R_\sim$ (depends on $\sim$ only), and the *maximal filtered relation* $R_{\sim,\Gamma}$ (depends on $\sim$ and Γ).

For a finite Γ, and a model M of the logic L, the goal is:

(1) to find a finite index refinement $\sim$ of $\sim_\Gamma$,

(2) and then to find $\widehat{R}$ between $R_\sim$ and $R_{\sim,\Gamma}$ which gives a model $\widehat{M}$ of L.

A class $\mathcal{F}$ of frames *admits filtration* if, for every finite Sub-closed set of formulas Γ and every $\mathcal{F}$-model M, there exists a finite filtration $\widehat{M}$ of M through Γ based on a frame in $\mathcal{F}$.

[Lemmon, 1966] If a class $\mathcal{F}$ admits filtration, then $\mathrm{Log}(\mathcal{F})$ has the fmp.

Moreover, we have:

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Formal definition of epifiltration (bonus track)

Put $a \sim_\Gamma b$ iff $\forall\psi \in \Gamma\,(M, a \models \psi \Leftrightarrow M, b \models \psi)$

Let Γ be a subformula-closed set of formulas, $M = (X, R, \theta)$ a model. An *(epi)filtration of M through Γ* is a model $\widehat{M} = (\widehat{X}, \widehat{R}, \widehat{\theta})$ such that:

(a) $\widehat{X} = X/{\sim}$ for some $\sim$ that refines $\sim_\Gamma$;

(b) $\widehat{M}, [a] \models p$ iff $M, a \models p$ for variables $p \in \Gamma$;

(c) $R_\sim \subseteq \widehat{R} \subseteq R_{\sim,\Gamma}$, where

$[a]\, R_\sim\, [b]$ iff $\exists a' \sim a\ \exists b' \sim b\ (a'\, R\, b')$,

$[a]\, R_{\sim,\Gamma}\, [b]$ iff $\Diamond\varphi \in \Gamma\ \&\ M, b \models \varphi \Rightarrow M, a \models \Diamond\varphi$

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The most standard application of the method:

$$\sim \text{ is } \sim_\Gamma.$$

For example, for the case of transitive frames, the transitive closure of the minimal filtered relation $R_{\sim_\Gamma}$ gives a filtration.

In general, letting the equivalence $\sim$ be finer than $\sim_\Gamma$ gives much more flexibility.

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Recall: L_m is the logic of frames s.t. $R^{m+\mathbf{1}} \subseteq R^{\mathbf{0}} \cup R$.

Theorem [Kudinov & Sh]

The class of L_m-frames admits filtration.

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The proof is based on the following two components:

I. Local tabularity of the logic of L_m-clusters [Shehtman & Sh, 2016]. This is used to define $\sim$. (Recall that a logic L is *locally tabular*, if, for every $k < \omega$, L contains only a finite number of pairwise nonequivalent formulas in a given k variables.)

II. Existence of the L_m*-closure*, which will be applied to the minimal filtered relation to define $\widehat{R}$

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Let (X, R) be a frame. By recursion on n, set:

$$R^{[m]}_{\mathbf{0}} = R, \quad R^{[m]}_{n+\mathbf{1}} = \left(\bigcup_{i\leq n} R^{[m]}_i\right)^{m+\mathbf{1}} \setminus Id_X.$$

Put

$$R^{[m]} = \bigcup_{n\in\omega} R^{[m]}_n.$$

$(X, R^{[m]})$ is called the L_m*-closure of* (X, R).

Proposition. $R^{[m]}$ is the smallest relation S on X s.t. S contains R and (X, S) validates L_m.

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II. Existence of the ***L_m-closure***, which will be applied to the minimal filtered relation to define $\widehat{R}$

Problem 1. Let $\mathcal{F}$ be a modally definable class of frames, $\mathcal{C}$ the class of clusters occurring in frames in $\mathcal{F}$. Assume that:

> $\mathrm{Log}(\mathcal{C})$ is locally tabular, and
>
> for any frame (X, R) there exists the smallest relation $R^{\mathcal{F}}$ containing R such that $(X, R^{\mathcal{F}}) \in \mathcal{F}$

Does $\mathcal{F}$ admit filtration?

Let $M = (X, R, \theta)$ and $M_\mathbf{0} = (X_\mathbf{0}, R_\mathbf{0}, \theta_\mathbf{0})$ be models, $X_\mathbf{0} \subseteq X$, $R_\mathbf{0} \subseteq R$, and $\theta_\mathbf{0}(p) = \theta(p) \cap X_\mathbf{0}$ for variables. Let Γ be a set of modal formulas closed under taking subfromulas. $M_\mathbf{0}$ is called a *selective filtration of M through Γ*, if for all $\Diamond\psi \in \Gamma$, $a \in X_\mathbf{0}$,

$$M, a \vDash \Diamond\psi \;\Rightarrow \exists b\,(aR_\mathbf{0}b \,\&\, M, b \vDash \psi).$$

Lemma. $\forall\psi \in \Gamma\,(M, a \vDash \psi \text{ iff } M_\mathbf{0}, a \vDash \psi)$

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Lemma. $\forall \psi \in \Gamma\,(M, a \vDash \psi$ iff $M_0, a \vDash \psi)$

Selective filtrations are very effective when combined with *maximality property* in canonical models.

In a transitive canonical frame, every non-empty definable subset has a maximal element [Fine, 85]. In fact:

Maximality lemma. Suppose that $\mathsf{F} = (X, R)$ is the canonical frame of a pretransitive L, Ψ is a set of formulas. If $\{a \in X \mid \Psi \subseteq a\}$ is non-empty, then it has a maximal element (with respect to the preorder R^*).

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Maximality + selective filtrations were used in [Fine, 85] to prove the FMP of subframe transitive logics. These results transfer to extensions of $\mathrm{wK4}$ ($m = 1$) [G. Bezhanishvili & Ghilardi & Jibladze, 2011] (algebraic methods).

Theorem [Kudinov&Sh]

Let L be a subframe canonical extension of L_m. Then it has the finite model property. Moreover, the size of a countermodel is exponential, and its height is bounded by m times the length of the formula.

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It is immediate that all logics L_m are PSpace-hard [Ladner, 77]. For the logic $\mathrm{wK4}$ ($m = 1$), the PSpace upper bound is known [Sh, 2022].

Problem 2. What is the complexity of the logics L_m for $m > 1$?

We conjecture that they all are PSpace-complete.

Let $M = (X, R, \theta)$ and $M_0 = (X_0, R_0, \theta_0)$ be models, $X_0 \subseteq X$, $R_0 \subseteq R$, and $\theta_0(p) = \theta(p) \cap X_0$ for variables. Let Γ be a set of modal formulas closed under taking subfromulas. M_0 is called a *selective filtration of M through Γ*, if for all $\Diamond\psi \in \Gamma$, $a \in X_0$,

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Very recently, the finite model property was announced for a family of m-transitive subframe logics that are weaker than L_m [Dvorkin, 24].

Problem 3. Do all Kripke complete subframe pretransitive logics have the FMP?

Let $M = (X, R, \theta)$ and $M_0 = (X_0, R_0, \theta_0)$ be models, $X_0 \subseteq X$, $R_0 \subseteq R$, and $\theta_0(p) = \theta(p) \cap X_0$ for variables. Let Γ be a set of modal formulas closed under taking subfromulas. M_0 is called a *selective filtration of M through Γ*, if for all $\Diamond\psi \in \Gamma$, $a \in X_0$,

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