# On expressibility and axiomatization of modal logics of distances

Ilya Shapirovsky

Joint work with Gabriel Agnew, Uzias Gutierrez-Hougardy, John Harding, Hannah Himelright, Jackson West, and Andrew Meléndrez Zerwekh on the project *"Research Training Group in Logic and Its Application 2023 – 2024"*

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In this talk I will announce recent developments, obtained by our group in the project *"Research Training Group in Logic and Its Application"*, 2023 – 2024.

- ➥ New theoretical results [Gabriel Agnew, Uzias Gutierrez-Hougardy, John Harding, Jackson West, Sh].
- ➥ Related software tools [Hannah Himelright and Andrew Meléndrez Zerwekh ].

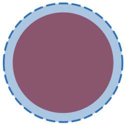
Let $X$ be a metric space. For instance, $X$ can be the real line $\mathbb{R}$, or the usual Euclidean plane $\mathbb{R}^2$, or any other set equipped with a distance function.

Two points are said to be *close*, if they are at a distance less than 1.

For a set $Y \subseteq X$, let $\langle c \rangle Y$ be the set of points $z$ such that each $z$ is close to some point in $Y$:

$$\langle c \rangle Y = \{z \mid d(z, y) < 1 \text{ for some } y \in Y\}$$

$Y$ is the red closed ball,
$\langle c \rangle Y$ is the blue open ball:



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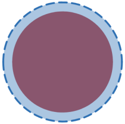

*Modal terms* are build using the operations of union, intersection, complement, and the operation $\langle \mathrm{c} \rangle$.
The *modal logic of closeness* $\mathrm{Log}_{\mathrm{c}}(X)$ is the set of all valid in $X$ statements $T \subseteq S$, where $T, S$ are modal terms.
Easy examples: $Y \subseteq \langle \mathrm{c} \rangle Y$ is true in all $\mathbb{R}^n$; $\langle \mathrm{c} \rangle \langle \mathrm{c} \rangle Y \subseteq \langle \mathrm{c} \rangle Y$ is not valid in unbounded spaces.

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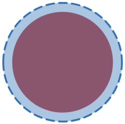

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It is less trivial to check that

we have in $\mathbb{R}$, but not in $\mathbb{R}^2$: $$\langle c \rangle Y_1 \cap \langle c \rangle Y_2 \cap \langle c \rangle Y_3 \subseteq \bigcup_{1 \leq i \neq j \leq 3} \langle c \rangle (Y_i \cap \langle c \rangle Y_j)$$

we have in $\mathbb{R}^2$, but not in $\mathbb{R}^3$: $$\langle c \rangle Y_1 \cap \ldots \cap \langle c \rangle Y_6 \subseteq \bigcup_{1 \leq i \neq j \leq 6} \langle c \rangle (Y_i \cap \langle c \rangle Y_j)$$

In fact, we have:

**Theorem (Agnew, Gutierrez-Hougardy, Harding, West, Sh)**
$\mathrm{Log}_c(\mathbb{R}) \supsetneq \mathrm{Log}_c(\mathbb{R}^2) \supsetneq \mathrm{Log}_c(\mathbb{R}^3) \supsetneq \mathrm{Log}_c(\mathbb{R}^4) \supsetneq \ldots$

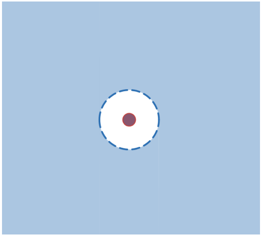
More details will be given in Gabriel's talk in a few minutes.

Two points are said to be *far*, if they are at a distance greater than 1.

Now for a set $Y \subseteq X$, let $\langle \mathrm{f} \rangle Y$ be the set of points $z$ such that each $z$ is far from some point in $Y$:

$$\langle \mathrm{f} \rangle Y = \{z \mid d(z, y) > 1 \text{ for some } y \in Y\}$$

$Y$ is the small red closed ball,
$\langle \mathrm{f} \rangle Y$ is the blue area:



The corresponding statements form $\mathrm{Log}_{\mathrm{f}}(X)$, the *modal logic of farness*.
Easy examples: $Y \subseteq \langle \mathrm{f} \rangle Y$ is not valid in $\mathbb{R}^n$, while $Y \subseteq \langle \mathrm{f} \rangle \langle \mathrm{f} \rangle Y$ is valid.

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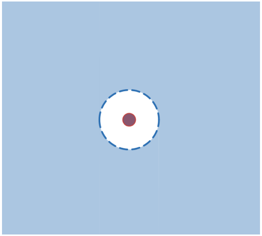

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**Theorem (Agnew, Gutierrez-Hougardy, Harding, West, Sh)**
For every dimension $n$, there exists $k(n)$ such that for all $m > k(n)$, $\mathrm{Log}_{\mathrm{f}}(\mathbb{R}^n) \neq \mathrm{Log}_{\mathrm{f}}(\mathbb{R}^m)$.

More details will be given by Uzias in his talk.

In a general setting, *modal logic* is defined for every relational structure $X$ (*frame*): for a binary relation $R$ on $X$, put $\Diamond_R Y = \{z \mid zRy \text{ for some } y \in Y\}$. The set $\mathrm{Log}\, X$ of valid in $X$ modal statements is called the *modal logic of* $X$.

*Axiomatization problem:* Describe a set $\Psi$ of formulas such that $\mathrm{Log}\, X$ is the set of logical consequences of $\Psi$. In this case, $\Psi$ is called a *complete axiomatization* of $\mathrm{Log}\, X$.

*Finite axiomatizability:* Does the logic have a finite complete axiomatization?

*Decidability:* Decidability of $\mathrm{Log}\, X$ means that there exists an algorithm correctly deciding which statements are valid in $X$, and which are not.

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Modal logics of relations induced by distance in a metric space:

**O. Kutz, H. Sturm, N.-Y. Suzuki, F. Wolter, and M. Zakharyaschev. *Axiomatizing distance logics*, 2002.**

**O. Kutz, F. Wolter, H. Sturm, N.-Y. Suzuki, and M. Zakharyaschev. *Logics of metric spaces*, 2003.**

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[Agnew, Gutierrez-Hougardy, Harding, West, Sh]:

1. A complete axiomatization was found for the logic of farness on the class $\mathrm{Unb}$ of unbounded metric spaces.
2. This logic and all $\mathrm{Log_f}\, \mathbb{R}^n$ are not finitely axiomatizable.
3. A finite complete axiomatization was found for the logic of farness on the class $\mathrm{Ult}$ of ultrametric spaces.
4. The logic of farness on $\mathrm{Unb}$ and the logic of farness on $\mathrm{Ult}$ are decidable.

More details will be given by Jackson in his talk.

The finite axiomatizability problem is closely related to the following property:

Two structures $F$ and $G$ are said to be *m-equivalent*, if they are indistinguishable by modal formulas containing at most $m$ propositional variables. In symbols: $F \sim_m G$.

*Lemma.* Let $L$ be a logic. Assume that for every $m$ there are structures $F_m$, $G_m$ such that $L$ is included in $\mathrm{Log}\, F_m$, $L$ is not included in $\mathrm{Log}\, G_m$, and

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It can be quite non-trivial to "manually" recognize $m$-equivalence, even for a pair of very small structures. However, on finite frames, this problem reduces to analyzing a finite family of quotient-frames, and so is decidable.

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# Thank you!